1. Experiments show that rubber-like materials have specific internal energy $\varepsilon(\theta)$ and heat capacity $c(\theta)$ that are both essentially *independent* of strain **C**

(a) Show that the heat capacity of an elastic material (regardless of the form of ε) is related to the specific entropy by

$$c = \theta \frac{\partial s}{\partial \theta}$$

(b) Hence, show that the entropy for rubber-like materials must have the separable form
$$s(\mathbf{C},\theta) = g(\mathbf{C}) + h(\theta)$$

(c) As a specific example, consider an incompressible, isotropic material of this kind, for which g is only a function of $\bar{I}_1 = \bar{B}_{kk}$ with $\bar{B}_{ij} = B_{ij} / J^{2/3}$ Show that Cauchy stress in such a material is given by

$$\sigma_{ij} = -2\rho_0\theta \frac{dg}{dI_1}\left(B_{ij} - \frac{1}{3}\bar{I}_1\delta_{ij} \right) + p\delta_{ij}$$

(d) Consider the simple shear deformation shown in the figure. Show that the shear stress in the solid is related to the shear strain $\gamma = \tan\theta$ by

$$\sigma_{12} = \mu\gamma \qquad \mu = -2\rho_0\theta \frac{dg}{dI_1}$$
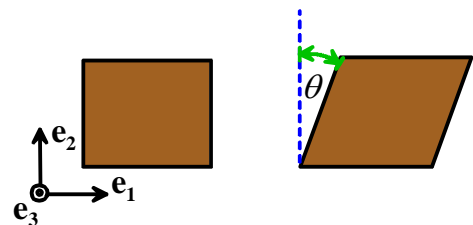
(The generalized shear modulus must satisfy $\mu > 0$)

(e) Show that the uniaxial Cauchy stress-strain response of the material is given by
$$\sigma_{11} = \mu(\lambda^2 - 1/\lambda)$$
(the solid is loaded in uniaxial tension parallel to $\mathbf{e}_1$, and $\lambda = l/L$ is the principal stretch parallel to the loading axis, with l the deformed and L the underformed length of the bar, respectively)

(f) Suppose that a bar of this rubber-like solid is loaded in uniaxial tension at a constant stress. How does the length of the bar change when its temperature is increased?

(g) Suppose that the bar is stretched quasi-statically at constant temperature, with normalized extension rate $\dot{\lambda} = \dfrac{1}{L}\dfrac{dl}{dt}$ (neglect body forces). Show that the heat flow per unit volume into the bar is

$$Q = -\mu\left(\lambda^2 - \frac{1}{\lambda}\right)\frac{\dot{\lambda}}{\lambda}$$

(i.e. when stretched, the bar gives off heat)

2. Derive the stress-strain relations for an incompressible, Neo-Hookean material subjected to
  (a) Uniaxial tension
  (b) Equibiaxial tension
  (c) Pure shear
  Derive expressions for the Cauchy stress, the Nominal stress, and the Material stress tensors (the solutions for nominal stress are listed in the notes). You should use the following procedure: (i) assume that the specimen experiences the length changes listed in the table in the notes ; (ii) use the stress-stretch relations to compute the Cauchy stress, leaving the hydrostatic part of the stress p as an unknown; (iii) Determine the hydrostatic stress from the boundary conditions (e.g. for uniaxial tensile parallel to $\mathbf{e}_1$ you know $\sigma_{22} = \sigma_{33} = 0$; for equibiaxial tension or pure shear in the $\mathbf{e}_1, \mathbf{e}_2$ plane you know that $\sigma_{33} = 0$)

Undeformed   Deformed
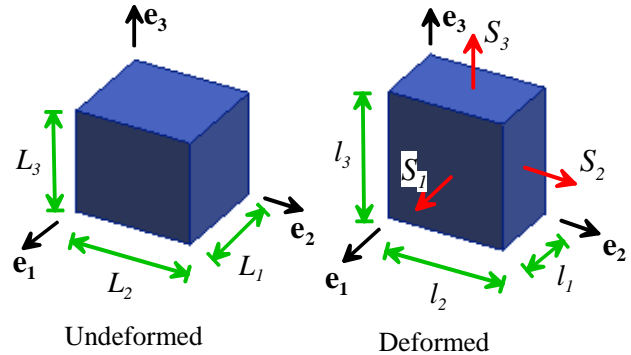
3. In a model experiment intended to duplicate the propulsion mechanism of the lysteria bacterium, a spherical bead with radius a is coated with an enzyme known as an "Arp2/3 activator." When suspended in a solution of actin, the enzyme causes the actin to polymerize at the surface of the bead. The polymerization reaction causes a spherical gel of a dense actin network to form around the bead. New gel is continuously formed at the bead/gel interface, forcing the rest of the gel to expand radially around the bead. The actin gel is a long-chain polymer and consequently can be idealized as a rubber-like incompressible neo-Hookean material. Experiments show that after reaching a critical radius the actin gel loses spherical symmetry and occasionally will fracture. Stresses in the actin network are believed to drive both processes. In this problem you will calculate the stress state in the growing, spherical, actin gel.
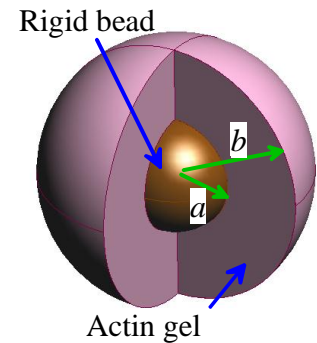
(a) Note that this is an unusual boundary value problem in solid mechanics, because a compatible reference configuration cannot be identified for the solid. Nevertheless, it is possible to write down a deformation gradient field that characterizes the change in shape of infinitesimal volume elements in the gel. To this end: (i) write down the length of a circumferential line at the surface of the bead; (ii) write down the length of a circumferential line at radius r in the gel; (iii) use these results, together with the incompressibility condition, to write down the deformation gradient characterizing the shape change of a material element that has been displaced from $r=a$ to a general position r. Assume that the bead is rigid, and that the deformation is spherically symmetric.

(b) Suppose that new actin polymer is generated at volumetric rate $\dot{V}$. Use the incompressibility condition to write down the velocity field in the actin gel in terms of $\dot{V}$, a and r (think about the volume of material crossing a radial line per unit time)

(c) Calculate the velocity gradient $\mathbf{v} \otimes \nabla$ in the gel (i) by direct differentiation of (b) and (ii) by using the results of (a). Show that the results are consistent.

(d) Calculate the components of the left Cauchy-Green deformation tensor field and hence write down an expression for the Cauchy stress field in the solid, in terms of an indeterminate hydrostatic pressure.

(e) Use the equilibrium equations and boundary condition to calculate the full Cauchy stress distribution in the bead. Assume that the outer surface of the gel (at $r=b$) is traction free.

4. A compressible, neo-Hookean solid has a stress-right C-G strain relation given by

$$\sigma_{ij} = \frac{\mu_1}{J^{5/3}} \left(B_{ij} - \frac{1}{3} B_{kk} \delta_{ij} \right) + K_1 (J-1) \delta_{ij}$$

Suppose that a solid consisting of such a material is first subjected to a deformation characterized by $F_{ij}^0, J_0 B_{ij}^0$, inducing a stress σ_{ij}^0. This deformation maps a material particle at position X_i in the reference configuration to position y_i in the deformed solid. The solid is then subjected to a further small deformation that induces an additional displacement distribution Δu_i in the material. Let

$$\Delta \varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial \Delta u_i}{\partial y_j} + \frac{\partial \Delta u_j}{\partial y_i} \right)$$

denote the increment of infinitesimal strain associated with this displacement, and expand the stress as a Taylor series in strain as

$$\sigma_{ij} = \sigma_{ij}^0 + C_{ijkl} \Delta \varepsilon_{kl} + O(\Delta \varepsilon_{kl})^2$$

Show that the tangent modulus for this deformation is

$$C_{ijkl} = \frac{\mu_1}{J_0^{5/3}} \left(B_{jl}^0 \delta_{ik} + B_{ik}^0 \delta_{jl} - \frac{2}{3} B_{kl}^0 \delta_{ij} \right) - \frac{5}{3} \frac{\mu_1}{J^{5/3}} \left(B_{ij}^0 - \frac{1}{3} B_{nn}^0 \delta_{ij} \right) \delta_{kl} + K \delta_{kl} \delta_{ij}$$

(Hint: note, eg, that the Jacobian after the incremental deformation can be approximated as

$$J \approx J_0 \left(1 + \frac{\partial \Delta u_m}{\partial y_m} \right)$$

5. A solid, spherical nuclear fuel pellet with outer radius a is subjected to a uniform internal distribution of heat due to a nuclear reaction. The heating induces a steady-state temperature field

$$T(r) = (T_a - T_0) \frac{r^2}{a^2} + T_0$$

where T_0 and T_a are the temperatures at the center and outer surface of the pellet, respectively. Assume that the pellet can be idealized as a linear elastic solid with Young's modulus E, Poisson's ratio v and thermal expansion coefficient α. Calculate the distribution of stress in the pellet.