School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 1: Index Notation; basic tensor operations
Solutions

1. Which of the following equations are valid expressions using index notation? If you decide an expression is invalid, state which rule is violated.

   (a) $S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}$ (b) (c) $\in_{ijk}\in_{kkj} = 0$ (d) $\rho\dfrac{\partial u_i}{\partial t} = \dfrac{\partial}{\partial x_j} C_{ijkl}\dfrac{\partial u_k}{\partial x_l}$

   (a) – OK. (b) (c) – no – the index k is repeated three times. (d) OK.

   [3 points]

2. Let $R = \sqrt{x_k x_k}$. Calculate $\dfrac{\partial \log(R)}{\partial x_i}$ and $\dfrac{\partial^2 \log(R)}{\partial x_i \partial x_i}$.

   Recall that $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$ and use the chain rule.

$$\frac{\partial \log(R)}{\partial x_i} = \frac{1}{R}\frac{1}{2R}\frac{\partial (x_k x_k)}{\partial x_i} = \frac{1}{2R^2}\left(\delta_{ki}x_k + x_k\delta_{ik}\right) = \frac{x_i}{R^2}$$

$$\frac{\partial^2 R}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{R^2} + x_i\frac{\partial}{\partial x_j}\left(\frac{1}{R^2}\right) = \frac{\delta_{ij}}{R^2} - 2\frac{x_i}{R^3}\frac{\partial R}{\partial x_j} = \frac{\delta_{ij}}{R^2} - 2\frac{x_i x_j}{R^4}$$

$$\Rightarrow \frac{\partial^2 R}{\partial x_i \partial x_i} = \frac{3}{R^2} - \frac{2}{R^2} = \frac{1}{R^2}$$

[2 POINTS]

3. Verify that (i.e. use the index notation rules to show that $S^{-1}_{ji}S_{im} = \delta_{jm}$)

$$S^{-1}_{ji}S_{im} = \frac{1}{2\det(\mathbf{S})}\in_{ipq}\in_{jkl} S_{pk}S_{ql}S_{im} = \frac{1}{2\det(\mathbf{S})}\in_{ipq}\in_{jkl} S_{pk}S_{ql}S_{im}$$

   Recall the identity $\in_{ijk} S_{il}S_{jm}S_{kn} = \in_{lmn}\det(\mathbf{S})$

   This gives $\dfrac{1}{2\det(\mathbf{S})}\in_{ipq}\in_{jkl} S_{pk}S_{ql}S_{im} = \dfrac{1}{2\det(\mathbf{S})}\in_{jkl}\in_{mkl}\det(\mathbf{S}) = \dfrac{1}{2}(\delta_{kk}\delta_{jm} - \delta_{jk}\delta_{mk}) = \delta_{jm}$

   [3 POINTS]

4. Use index notation rules to show that $\nabla\times\nabla\times\mathbf{u} = \nabla\nabla\cdot\mathbf{u} - \nabla^2\mathbf{u}$

Expand the LHS in index notation

$$\in_{ijk} \frac{\partial}{\partial x_j} \in_{klm} \frac{\partial}{\partial x_l} u_m = \in_{kij} \in_{klm} \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l} u_m = \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l} u_m$$
$$= \frac{\partial}{\partial x_m}\frac{\partial}{\partial x_i} u_m - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} u_i$$

[3 POINTS]

5. Let $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ be a (not necessarily Cartesian) basis in $\mathbb{R}^3$, and let $g_{ij} = \mathbf{m}_i \cdot \mathbf{m}_j$, $\mathbf{T} = T_{ij}\mathbf{m}_i \otimes \mathbf{m}_j$ $\quad \mathbf{S} = S_{ij}\mathbf{m}_i \otimes \mathbf{m}_j$ $\quad \mathbf{U} = \mathbf{ST}$. Find an expression for the components of $\mathbf{U}$ in $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$.

$$\mathbf{ST} = \quad \mathbf{S} = \left(S_{ij}\mathbf{m}_i \otimes \mathbf{m}_j\right)\left(T_{kl}\mathbf{m}_k \otimes \mathbf{m}_l\right) = S_{ij}\left(\mathbf{m}_j \cdot \mathbf{m}_k\right)T_{kl}\mathbf{m}_i \otimes \mathbf{m}_l = S_{ij}g_{jk}T_{kl}\mathbf{m}_i \otimes \mathbf{m}_l$$

Hence $U_{il} = S_{ij}g_{jk}T_{kl}$

[2 POINTS]

6. Let $\mathbf{a}, \mathbf{b}$ be two (not necessarily orthogonal) unit vectors in $\mathbb{R}^3$. Find formulas for the eigenvalues and eigenvectors of $\mathbf{S} = \mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b}$, in terms of $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$. (Don't use the standard formulas to do this. You can write down one eigenvalue and eigenvector by inspection. This and the symmetry of **S** then tells you something about the direction of the other two eigenvectors. You can use that insight to find the remaining eigenvectors, and finally deduce the eigenvalues).

By inspection, one eigenvector (with a null eigenvalue) must be perpendicular to $\mathbf{a}, \mathbf{b}$. Thus $\mathbf{e}_3 = \mathbf{a} \times \mathbf{b}$, $\lambda_3 = 0$

Since **S** is symmetric, the eigenvectors must be orthogonal, which means that the other two eigenvectors must lie in the **a,b** plane. We can construct these as $\mathbf{e} = \mathbf{a} + \beta\mathbf{b}$, where β is to be determined. By definition

$$\mathbf{S}\left[\mathbf{a} + \beta\mathbf{b}\right] = \left[\mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b}\right]\left[\mathbf{a} + \beta\mathbf{b}\right] = \lambda\left[\mathbf{a} + \beta\mathbf{b}\right]$$

We can take the dot product of this with **a**,**b** to give

$$\mathbf{a} \cdot \left[\mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b}\right]\left[\mathbf{a} + \beta\mathbf{b}\right] = \lambda\mathbf{a} \cdot \left[\mathbf{a} + \beta\mathbf{b}\right]$$
$$\Rightarrow (1 + \beta\mathbf{a} \cdot \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}(\mathbf{a} \cdot \mathbf{b} + \beta) = \lambda(1 + \beta\mathbf{a} \cdot \mathbf{b})$$
$$\mathbf{b} \cdot \left[\mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b}\right]\left[\mathbf{a} + \beta\mathbf{b}\right] = \lambda\mathbf{b} \cdot \left[\mathbf{a} + \beta\mathbf{b}\right]$$
$$\Rightarrow \mathbf{a} \cdot \mathbf{b}(1 + \beta\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{b} + \beta) = \lambda(\mathbf{a} \cdot \mathbf{b} + \beta)$$

If we let $\gamma = \mathbf{a} \cdot \mathbf{b}$ this is two equations for β and λ

$$(1 + \beta\gamma) + \gamma(\gamma + \beta) = \lambda(1 + \beta\gamma)$$
$$\gamma(1 + \beta\gamma) + (\gamma + \beta) = \lambda(\gamma + \beta)$$

These are easily solved to see that

$$\beta = 1, \lambda = 1 + \gamma \qquad \beta = -1, \lambda = 1 - \gamma \qquad \gamma \neq 1$$

We can check (or with some insight construct these directly)

$$\left[\mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b}\right]\left[\mathbf{a} + \mathbf{b}\right] = \mathbf{a}(1 + \mathbf{a} \cdot \mathbf{b}) + \mathbf{b}(1 + \mathbf{a} \cdot \mathbf{b}) = (1 + \mathbf{a} \cdot \mathbf{b})\left[\mathbf{a} + \mathbf{b}\right]$$
$$\left[\mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b}\right]\left[\mathbf{a} - \mathbf{b}\right] = \mathbf{a}(1 - \mathbf{a} \cdot \mathbf{b}) + \mathbf{b}(\mathbf{a} \cdot \mathbf{b} - 1) = (1 - \mathbf{a} \cdot \mathbf{b})\left[\mathbf{a} - \mathbf{b}\right]$$

For the special case $\mathbf{a}\cdot\mathbf{b}=1$ there is a repeated eigenvector of 1. Any vector in the $\mathbf{a},\mathbf{b}$ plane is an eigenvector, but we can pick $\mathbf{a}\ (\lambda=1)$ $\mathbf{b}(\lambda=1)$ $\mathbf{a}\times\mathbf{b}\ (\lambda=0)$ as three mutually perpendicular eigenvectors with their corresponding eigenvalues.

[5 POINTS]

7. Let $\mathbf{R}$ be a proper orthogonal tensor (det(**R**)>**0)**. Let I_1, I_2, I_3 be the three invariants of **R** defined in class. Show that $I_1 = I_2$ (but do this without any index notation manipulations).

   Recall that $I_3 = \det(\mathbf{R}) = 1$ **.** The characteristic equation is $\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$. But we know that this must be satisfied for $\lambda = 1$ which can only be the case if $I_1 = I_2$.

[2 POINTS]

8. Let **n** be the dual vector of a skew tensor $\mathbf{W}$ **.** What is $\mathbf{Wn}$?

   Since **n** is the dual vector we have that $\mathbf{Wn} = \mathbf{n}\times\mathbf{n} = \mathbf{0}$

[1 POINT]

9. Let **W** be a skew tensor. Show that $\mathbf{I}+\mathbf{W}$ is nonsingular and $(\mathbf{I}-\mathbf{W})(\mathbf{I}+\mathbf{W})^{-1}$ is orthogonal.

   Suppose that $\mathbf{I}+\mathbf{W}$ is singular. Then there exists some vector **x** that satisfies $(\mathbf{I}+\mathbf{W})\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{Wx} = -\mathbf{x}$. But this would require $\mathbf{x}\cdot\mathbf{Wx} = -\mathbf{x}\cdot\mathbf{x}$ and we have a contradiction, since $\mathbf{Wx}$ must be perpendicular to **x.**

$$\left[(\mathbf{I}-\mathbf{W})(\mathbf{I}+\mathbf{W})^{-1}\right]^T(\mathbf{I}-\mathbf{W})(\mathbf{I}+\mathbf{W})^{-1} = (\mathbf{I}+\mathbf{W})^{-T}(\mathbf{I}-\mathbf{W})^T(\mathbf{I}-\mathbf{W})(\mathbf{I}+\mathbf{W})^{-1}$$

Now $= (\mathbf{I}-\mathbf{W})^{-1}(\mathbf{I}+\mathbf{W})(\mathbf{I}-\mathbf{W})(\mathbf{I}+\mathbf{W})^{-1} = (\mathbf{I}-\mathbf{W})^{-1}(\mathbf{I}-\mathbf{WW})(\mathbf{I}+\mathbf{W})^{-1}$

$$= (\mathbf{I}-\mathbf{W})^{-1}(\mathbf{I}-\mathbf{W})(\mathbf{I}+\mathbf{W})(\mathbf{I}+\mathbf{W})^{-1} = \mathbf{I}$$

[5 POINTS]

10. Let **S** be a non-singular second order tensor with invariants I_1, I_2, I_3 . Show that $\mathbf{S}^{-1} = (\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{I})/I_3$

This follows directly from the Cayley-Hamilton theorem

[2 POINTS]

11. Let **S** be symmetric and **W** skew. Calculate $\mathbf{S}:\mathbf{W}$

$$S_{ij}W_{ij} = S_{ji}W_{ij} = -S_{ji}W_{ji} = -S_{ij}W_{ij} \Rightarrow S_{ij}W_{ij} = 0$$

[2 POINTS]