School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 5: Thermodynamics and Constitutive Equations
Due Wednesday November 2, 2016

1. Define the *expended power* of external forces acting on a deformable solid (which could be a sub-volume within a larger body) by

$$W_{\exp} = \int_S \mathbf{t}\cdot\mathbf{v} + \int_R \rho\mathbf{b}\cdot\mathbf{v} - \frac{d}{dt}\int_R \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}$$

Show that the expended power is zero for any rigid velocity field of the form

$$\mathbf{v}(\mathbf{y},t) = \mathbf{v}_0(t) + \boldsymbol{\omega}(t)\times(\mathbf{y}-\mathbf{y}_0)$$

where $\mathbf{v}_0(t),\ \boldsymbol{\omega}(t)$ are vector valued functions of time (but independent of position).

It is helpful to re-write the time derivative of kinetic energy in terms of accelerations

$$\frac{d}{dt}\int_R \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} = \frac{d}{dt}\int_{R_0} \frac{1}{2}\rho_0\mathbf{v}\cdot\mathbf{v}dV = \int_{R_0} \rho_0\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}dV = \int_R \rho\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}dV$$

Hence

$$W_{\exp} = \int_S \mathbf{t}\cdot\mathbf{v} + \int_R \rho\left(\mathbf{b}-\frac{d\mathbf{v}}{dt}\right)\cdot\mathbf{v}dV$$

It is also convenient to re-write the velocity field as

$$\mathbf{v}(\mathbf{y},t) = \mathbf{v}_0(t) + \mathbf{W}(\mathbf{y}-\mathbf{y}_0)$$

where **W** is a skew tensor that has $\boldsymbol{\omega}(t)$ as its dual vector.

2. It is helpful to have versions of the first and second laws of thermodynamics in terms of quantities defined on the reference configuration. To this end, define :

- Reference mass density ρ_0
- Nominal stress and deformation gradient $\mathbf{S}$, $\mathbf{F}$
- Material stress and Lagrange strain rate $\boldsymbol{\Sigma}$, $\mathbf{E}$
- Referential heat flux $\boldsymbol{\Theta} = J\mathbf{F}^{-1}\mathbf{q}$

With these definitions, follow the same procedure used in class to derive the spatial laws of thermodynamics (but now express the classical laws as integrals over the reference configuration) to show the following identities:

$$\rho_0 \left.\frac{\partial \varepsilon}{\partial t}\right|_{\mathbf{x}=const} = S_{ij}\frac{dF_{ji}}{dt} - \frac{\partial Q_i}{\partial x_i} + J\eta$$

$$\rho_0 \left.\frac{\partial \varepsilon}{\partial t}\right|_{\mathbf{x}=const} = \Sigma_{ij}\frac{dE_{ij}}{dt} - \frac{\partial Q_i}{\partial x_i} + J\eta$$

$$S_{ij}\frac{dF_{ji}}{dt} - \frac{1}{\theta}Q_i\frac{\partial \theta}{\partial x_i} - \rho_0\left(\frac{\partial \psi}{\partial t} + s\frac{\partial \theta}{\partial t}\right) \ge 0$$

$$\Sigma_{ij}\frac{dE_{ij}}{dt} - \frac{1}{\theta}Q_i\frac{\partial \theta}{\partial x_i} - \rho_0\left(\frac{\partial \psi}{\partial t} + s\frac{\partial \theta}{\partial t}\right) \ge 0$$

3. Starting with the local form of the second law of thermodynamics and mass conservation

$$\rho\left.\frac{\partial s}{\partial t}\right|_{\mathbf{x}=const} + \frac{\partial(q_i/\theta)}{\partial y_i} - \frac{q}{\theta} \ge 0 \qquad \left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}} + \frac{\partial \rho v_i}{\partial y_i} = 0$$

(the symbols have their usual meaning), derive the statement of the second law for a control volume

$$\frac{\partial}{\partial t}\int_R \rho s dV + \int_B \rho s(\mathbf{v}\cdot\mathbf{n})dA + \int_B \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}dA - \int_R \frac{q}{\theta}dV \ge 0$$

4. Determine how the following quantities transform under a change of observer

(i) The spatial heat flux vector $\mathbf{q}$ (recall that $\mathbf{q}\cdot\mathbf{n}dA$ gives the heat flux across a surface element with area dA and normal $\mathbf{n}$, and note that $\mathbf{n}$ is an objective vectors, and that the all observers must see the same heat flux….)

(ii) The referential heat flux vector $\boldsymbol{\Theta} = J\mathbf{F}^{-1}\mathbf{q}$

(iv) The infinitesimal strain tensor $\boldsymbol{\varepsilon} = \left[\nabla_{\mathbf{x}}\mathbf{u} + \left(\nabla_{\mathbf{x}}\mathbf{u}\right)^T\right]/2$

(iv) The spatial gradient of a scalar function of position in a deformed solid $\mathbf{g} = \nabla_{\mathbf{y}}\phi(\mathbf{y})$

(v) The material gradient of a function of particle position in a solid $\mathbf{G} = \nabla\phi(\mathbf{x})$