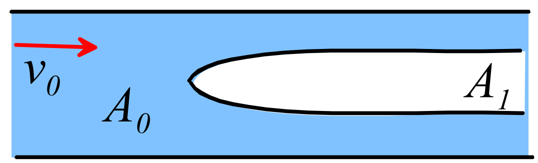
School of Engineering
Brown University

EN2210: Continuum Mechanics

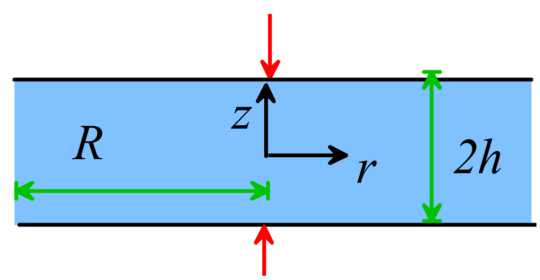
Homework 7: Fluid Mechanics
Due Wed Nov 16 2016

1. An ideal fluid with mass density ρ flows with velocity v_0 through a cylindrical tube with cross-sectional area A_0. A body extending downstream to infinity with cross-sectional area A_1 is located within the tube. Use the conservation equations for a fluid in control volume form to find an expression for the force exerted by the fluid on the body in the tube.



2. An incompressible Newtonian viscous fluid with viscosity η and density ρ occupies the region $y_2 < 0$. At time *t=0* the fluid has velocity distribution $v_1 = u_0 \exp(-y_2^2 / b^2)$ with all other velocity components zero. Gravity and pressure variations in the fluid may be neglected. By considering a velocity field of the form $v_1 = u_0 f(t) \exp(-y_2^2 f(t)^2 / b^2)$, calculate the variation of velocity in the fluid with position and time.

3. Two parallel circular plates with radius *R* are separated by an incompressible Newtonian fluid with viscosity η. The plates are a distance $2h$ apart and approach one another slowly with relative speed $2U$. The goal of this problem is to find an approximate solution for the velocity distribution in the fluid between the plates. Assume Stokes flow and neglect the acceleration, and assume a velocity field of the form $v_r = rf(z) \qquad v_z = g(z)$



3.1 Use the incompressibility condition to determine $f(z)$ in terms of $g(z)$

3.2 Hence, find an expression for the velocity gradient in terms of $g(z)$ and *r* (use cylindrical polar coordinates)

3.3 Write down *two* boundary conditions for $g(z)$ at $z = \pm h$

3.3 Show that for this problem the principle of virtual work reduces to

$$\int_0^R \int_{-h}^{h} 2\eta \nabla \mathbf{D} : \nabla \delta \tilde{\mathbf{v}} \; dz \; 2\pi r dr = 0 \qquad \forall \;\; \delta\tilde{\mathbf{v}} \; : \nabla\cdot\delta\tilde{\mathbf{v}} = 0$$

3.4 Hence, show that

$$\int_0^R dr \int_{-h}^{h} 2\pi r 2\eta \left[\left(\frac{3}{2}\frac{dg}{dz}\right)\left(\frac{d\delta g}{dz}\right) + \frac{r^2}{8}\left(\frac{d^2 g}{dz^2}\right)\left(\frac{d^2 \delta g}{dz^2}\right)\right] dz = 0$$

3.5 Deduce that

$$\frac{d^2 g}{dz^2} - \frac{R^2}{24}\frac{d^4 g}{dz^4} = 0$$

and hence determine *g(z)*.

3.6 Show that for $R >> h$ the solution can be approximated further by
$g(z) = U(z^3 / h^3 - 3z / h) / 2$

3.7 For the approximation in 3.6, calculate the pressure distribution in the fluid, and deduce an expression for the force required to squeeze the plates together (assume the integral of the pressure through the film thickness at $r=R$ vanishes). You can assume that the Stokes equation for a cylindrically symmetric flow is

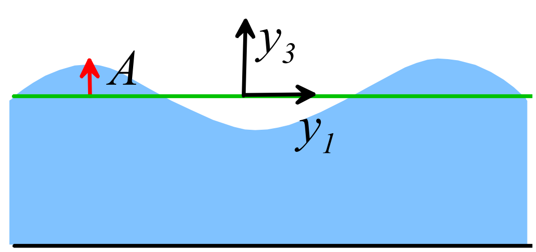
$$-\frac{\partial p}{\partial r} + \eta\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2}\right) = 0$$

$$-\frac{\partial p}{\partial z} + \eta\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{\partial^2 v_z}{\partial z^2}\right) = 0$$

4. The potential

$$\Omega = \frac{\omega}{k} A \frac{\cosh(k(y_3 + h))}{\sinh(kh)} \sin(ky_1 - \omega t)$$

describes small amplitude surface waves in an ideal fluid with depth h .



4.1 Calculate the velocity distribution in the fluid

4.2 Show that the velocity at the surface of the fluid is consistent with a surface wave with a vertical displacement $w_3 = A\cos(ky_1 - \omega t)$. What are the trajectories of material particles in the fluid?

4.3 Find an expression for the pressure in the fluid (neglect the term of order A^2)

4.4 For small amplitude waves the boundary condition on the fluid surface can be expressed as

$$p(y_3 = 0) + \left.\frac{\partial p}{\partial y_3}\right|_{x_3=0} A\cos(ky_1 - \omega t) \approx 0$$

Find a condition relating k and ω (the dispersion relation) necessary to satisfy this condition (neglect the term of order A^2)

Water waves are a rich source of math problems: This paper is an example

5. By considering a flow potential of the form $\Omega = A\sin\omega t \sin ky_1$ calculate the natural frequencies of a 1-D air column with sound speed c_s , for the following boundary conditions:

(i) The end at $y_1 = 0$ is open (constant pressure); while the end at $y_1 = L$ is closed (zero velocity).

(ii) Both ends are open.

(iii) Estimate the length of a transverse flute whose lowest frequency is 246.9 Hz (both ends of a flute are open)