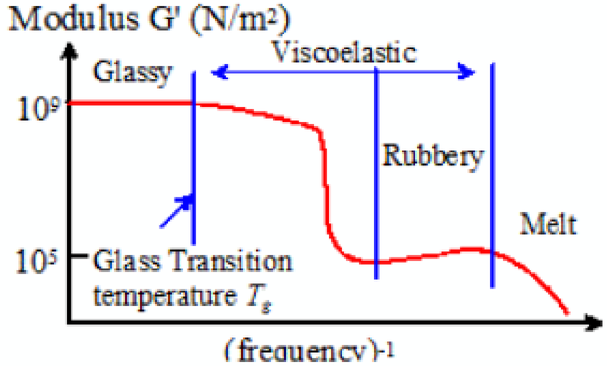
Review – constitutive models for hyperelastic materials

Goal: develop an exact stress-strain law that describes large deformations of elastomeric materials in the 'rubbery' regime



Assumptions:

- Local Action
- Perfectly Reversible
- History independent

$$\mathbf{C} = \mathbf{F}^T \mathbf{F}$$

We will show that the constitutive law must have the following structure:

Specific internal energy $\varepsilon = \hat{\varepsilon}(\mathbf{C}, \theta)$

Specific entropy $s = \hat{s}(\mathbf{C}, \theta)$

Specific Helmholtz free energy $\psi = \hat{\psi}(\mathbf{C}, \theta)$

Material stress response function $\mathbf{\Sigma} = \hat{\mathbf{\Sigma}}(\mathbf{C}, \theta)$

Material heat flux response function $\mathbf{Q} = \hat{\mathbf{Q}}(\mathbf{C}, \theta)$

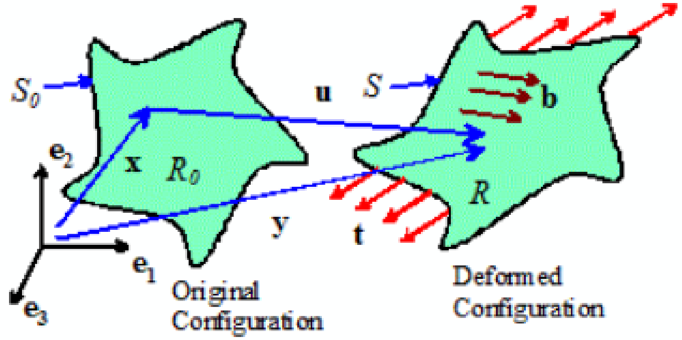
$$\Sigma_{ij} = 2\rho_0 \frac{\partial \hat{\psi}}{\partial C_{ij}} \qquad s = -\frac{\partial \hat{\psi}}{\partial \theta}$$

$$\frac{\partial \Sigma_{ij}}{\partial \theta} = -2\rho_0 \frac{\partial \hat{s}}{\partial C_{ij}} \qquad -\hat{\mathbf{Q}} \cdot \nabla \theta \geq 0$$

Notes: Other equivalent forms/relations exist
Other less general forms are also often used

Review

Deformation Gradient $\mathbf{F} = \nabla \mathbf{y} = \mathbf{I} + \nabla \mathbf{u} \qquad J = \det(\mathbf{F})$



Cauchy-Green Strains $\mathbf{C} = \mathbf{F}^T \mathbf{F}$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T$$

Nominal Stress $\mathbf{S} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma}$

Material Stress $\boldsymbol{\Sigma} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-T} \qquad \Sigma_{ij} = JF_{ik}^{-1} \sigma_{kl} F_{jl}^{-1}$

Referential Heat Flux $\mathbf{Q} = J\mathbf{F}^{-1} \cdot \mathbf{q}$

Dissipation Inequality

$$\boldsymbol{\Sigma} : \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{\theta} \mathbf{Q} \nabla \theta - \rho_0 \left(\frac{\partial \psi}{\partial t} + s \frac{\partial \theta}{\partial t} \right) \geq 0$$

$$\frac{1}{2} \boldsymbol{\Sigma} : \frac{\partial \mathbf{C}}{\partial t} - \frac{1}{\theta} \mathbf{Q} \nabla \theta - \rho_0 \left(\frac{\partial \psi}{\partial t} + s \frac{\partial \theta}{\partial t} \right) \geq 0$$

$$\mathbf{S} \cdot\cdot \frac{\partial \mathbf{F}}{\partial t} - \frac{1}{\theta} \mathbf{Q} \nabla \theta - \rho_0 \left(\frac{\partial \psi}{\partial t} + s \frac{\partial \theta}{\partial t} \right) \geq 0$$

Observer changes

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F}$$

$$\mathbf{B}^* = \mathbf{F}^* \mathbf{F}^{*T} = \mathbf{Q}\mathbf{B}\mathbf{Q}^T$$

$$\mathbf{C}^* = \mathbf{F}^{*T} \mathbf{F}^* = \mathbf{C}$$

$$\boldsymbol{\Sigma}^* = \boldsymbol{\Sigma}$$

General structure of constitutive equations

Assume Local Action; perfect reversibility; no history/rate dep.

Hence $\psi = \hat{\psi}(F, \theta)$ $\qquad \varepsilon = \hat{\varepsilon}(F, \theta)$

Frame Indifference

$$\psi^* = \psi = \hat{\psi}(QF, \theta) = \hat{\psi}(F, \theta) \quad \forall \; Q \in \text{orth}^+$$

Recall $F = R\sqrt{C}$ $\qquad F = RU \qquad U = \sqrt{C}$

$$\Rightarrow \hat{\psi}(QR\sqrt{C}, \theta) = \hat{\psi}(F, \theta)$$

This holds for $Q = R^T \Rightarrow$ $\boxed{\hat{\psi}(F, \theta) = \hat{\psi}(C, \theta)}$

This applies to all other functions as well

Dissipation Inequality

$$\frac{1}{2}\varepsilon : \frac{\partial c}{\partial t} - \frac{1}{\theta}\underline{Q}\cdot\nabla\theta - \rho_0\left(\frac{\partial\psi}{\partial t} + s\frac{\partial\theta}{\partial t}\right) \geq 0$$

$$\frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial c} : \frac{\partial c}{\partial t} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial t}$$

$$\left(\frac{1}{2}\varepsilon - \rho_0\frac{\partial\psi}{\partial c}\right) : \frac{\partial c}{\partial t} - \rho_0\left(\frac{\partial\psi}{\partial\theta} + s\right)\frac{\partial\theta}{\partial t} - \frac{1}{\theta}\underline{Q}\cdot\nabla\theta \geq 0$$

$$\Rightarrow \boxed{\varepsilon = 2\rho_0\frac{\partial\psi}{\partial c} \qquad s = -\frac{\partial\psi}{\partial\theta}}$$

Other equivalent forms

Note $\frac{1}{2} \Sigma : \frac{\partial C}{\partial t} = S \cdot\cdot \frac{\partial F}{\partial t}$

$\Rightarrow$ DI implies $S = \rho_0 \left(\frac{\partial \Psi}{\partial F}\right)^T = S_{ij} = \rho_0 \frac{\partial \Psi}{\partial F_{ji}}$

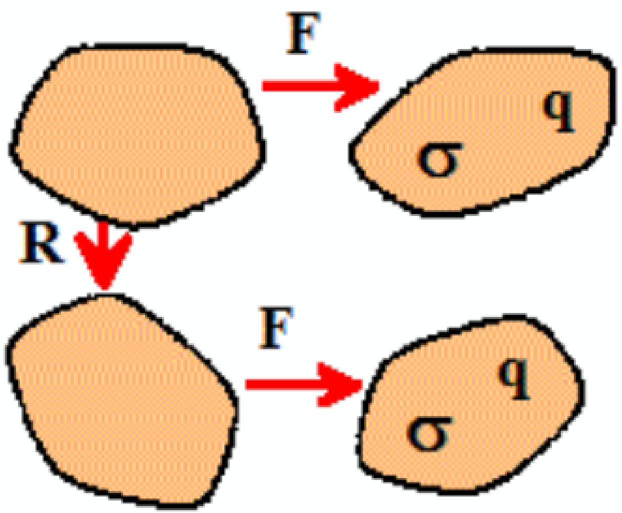
Also $\boxed{\sigma = \frac{1}{J} F \left(\frac{\partial \Psi}{\partial F}\right)^T}$

Natural Reference configuration

Note that $\hat{\Sigma}(I, \theta) \neq 0$ ie ref config is not stress free

A ref config with $\hat{\Sigma}(I, \theta) = 0$ is a "natural" ref config

Isotropic Materials



An isotropic material has ψ or $\hat{\Sigma}$ that are independent of orientation of material wrt principal strain directions

This requires $\hat{\psi}(FR, \theta) = \hat{\psi}(F, \theta)$

or $\hat{\psi}(F^T F, \theta) = \hat{\psi}(R^T F^T F R, \theta) = \hat{\psi}(F^T F, \theta)$

$\Rightarrow \hat{\psi}(R^T C R, \theta) = \hat{\psi}(C, \theta)$ holds $\forall$ orth$^+$ R

This means $\hat{\psi}$ must be an isotropic function

$\hat{\psi}$ must be a function of the invariants of C

Similarly $\hat{\Sigma}$ can only depend on invariants of C

Observations from experiment

For most elastomers : (1) Heat capacity c_v is independent of strain

(2) Internal energy ε is also independent of strain

Hence $\varepsilon = \psi + \theta s$ $c_v = \frac{\partial \varepsilon}{\partial \theta}$ $s = -\frac{\partial \psi}{\partial \theta}$

$$\Rightarrow c_v = \frac{\partial \varepsilon}{\partial \theta} = \cancel{\frac{\partial \psi}{\partial \theta}} - \cancel{\frac{\partial \psi}{\partial \theta}} + \theta \frac{\partial s}{\partial \theta}$$

Hence $s = \int \frac{c_v}{\theta} d\theta + g(C)$

Hence $\boxed{\psi = \theta g(C) + f(\theta)}$

Constitutive equations used in practice

Preliminaries: Recall "standard" invariants of C are (same as invariants of B)

$$I_1 = tr(B) = tr(C)$$

$$I_2 = \frac{1}{2}\left(I_1^2 - B:B\right) = \frac{1}{2}\left(I_1^2 - C:C\right)$$

$$I_3 = \det(B) = \det(C)$$

The volumetric and shear measures of strain do not decouple with this set

eg let $F = \beta I \qquad B = \beta^2 I = C$

$$\Rightarrow I_1 = 3\beta^2 \qquad I_2 = \frac{1}{2}\left(9\beta^4 - 3\beta^4\right) \qquad I_3 = \beta^6$$

Another set of invariants (used in ABAQUS) can be used

$$\bar{I}_1 = \frac{tr(B)}{J^{2/3}} = \frac{tr(C)}{J^{2/3}}$$

$$\bar{I}_2 = \frac{1}{2}\left(\bar{I}_1^2 - \frac{B:B}{J^{4/3}}\right) = \frac{1}{2}\left(\bar{I}_1^2 - \frac{C:C}{J^{4/3}}\right)$$

$$\bar{I}_3 = \bar{J}$$

For a volumetric deformation $F = \beta I$

$$\bar{I}_1 = 3 \qquad \bar{I}_2 = 3, \quad \bar{I}_3 = \beta^3 = J$$

Finally many material models use eigenvalues of B and C

Let $\lambda_1, \lambda_2, \lambda_3$ be eigenvalues of U

Let $\underline{b}^{(i)}$ $i = 1, 2, 3$ be principal directions of B

Now we can express $\rho_0 \psi = \underbrace{\theta \rho_0 g(\underline{c})}$ denote this by U

Can use $U(I_1, I_2, I_3) = \bar{U}(\bar{I}_1, \bar{I}_2, \bar{I}_3) = \hat{U}(\lambda_1, \lambda_2, \lambda_3)$