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330:542 — Information Theory and Coding — Spring 1997

Problem Set 4 Solutions

1. (a)

$$Q_n = C\prod_{i=1}^{n} X_i$$

$$H(X) = \sum_x p(x)log(\frac{1}{p(x)}) = \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{4}(2) = \frac{3}{2}$$

(b)

$$E(Q_n) = C\prod_{i=1}^{n} E(X_i)$$

$$E(X_i) = \frac{1}{2}(2) + \frac{1}{4}(\frac{1}{2}) + \frac{1}{4}(\frac{1}{4}) = 1 + \frac{1}{8} + \frac{1}{16} = 1\frac{3}{16}$$

Therefore,
$E(Q_n) = C(1\frac{3}{16})^n$
appears to be a good sheme since $E(Q_n) \to \infty$ with n
(ie $\alpha^n \to \infty$ if $\alpha > 1$)

(c)

$$E(logQ_n) = logC + \sum_{i=1}^{n} E(logx_i)$$

$E(logx_i) = \frac{1}{2}(1) + \frac{1}{4}(-1) + \frac{1}{4}(-2) = \frac{1}{2} - \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$
Therefore,
$E(logQ_n) = logC - \frac{n}{4}$
$E(logQ_n) \to -\infty$ as n $\to \infty \Rightarrow$ bad scheme $'(log0) = -\infty'$
Since log is an invertible function, each $Q_n$ corresponds to a unique $R_n$ thus the distributions will have the same terms.
$H(Q_n) = H(R_n) = nH(X) = \frac{3n}{2}$

(d) Tricky

$$R_n = \sum_{i=1}^{n} q_i$$

where $q_i = logX_i$
Therefore, $R_n$ is a sum of independent r.v's.
Limiting distribution of $R_n$ is Gaussian so lets look at mean and variance
$R_n = logC - \frac{n}{4}$
$\sigma^2_{R_n} = n\sigma^2_{log(x)}$
$\sigma_{R_n} = \sqrt{n}\sigma_{log(x)}$

$$\begin{aligned}\sigma^2_{log(x)} &= \tfrac{1}{4}(-2-(-\tfrac{1}{4}))^2 + \tfrac{1}{4}(-1+\tfrac{1}{4})^2 + \tfrac{1}{2}(1+\tfrac{1}{4})^2 \\ &= \tfrac{1}{4}(\tfrac{7}{4})^2 + \tfrac{1}{4}(\tfrac{3}{4})^2 + \tfrac{1}{2}(\tfrac{5}{4})^2)^2\end{aligned}$$

$$\lim_{n\to\infty} \frac{\sigma_{R_n}}{\overline{R_n}} = 0 \Rightarrow$$

relative width of distribution narrows with increasing mean
Therefore,
$Prob(\frac{|R_n - \overline{R_n}|}{\overline{R_n}} > \epsilon) \to 0$ with n $\to \infty$
Therefore,
all money lost with prob 1 $\to$ BAD SCHEME.

2. (a) $2^n$ source sequences $2^{\frac{n}{2}} = M$
   step by step $\Rightarrow \frac{logM}{n} = \frac{1}{2} \Rightarrow M = 2^{\frac{n}{2}}$

   (b) $P_0 = P_1 = \frac{1}{2} \Rightarrow$ H(X)=1 but we only have $2^{\frac{n}{2}}$ codewords. Prob of each source seq is $2^{-n} \Rightarrow$ no matter which source sequences are chosen the prob corresponding to the codewords is $(2_{-n})(2^{\frac{n}{2}}) = 2^{-\frac{n}{2}} \Rightarrow$ not really good odds on losing a limb.
   $Prob(default) = 1 - 2^{-\frac{n}{2}}$ no n will make this really small (except n=0 which is nonsensical)

   (c) Now we have $R = H(source) + \epsilon = \frac{logM}{n}$
   So now we need to be sensitive with our sequences
   <u>AEP time !!</u>
   Hand waving : Choose typical sequences! We know that we can find a typical set whose prob approaches 1 and whose size is less than $2^{n(H(X)+\epsilon)}$
   Therefore,
   $M \leq 2^{n(H(X)+\epsilon)}$ So set equal
   $M = 2^{n(H(X)+\epsilon)}$
   $\frac{logm}{n} = H(X) + \epsilon = R$
   So for sufficiently large n we can satisfy (or exceed) the rate criterion (ie. we can make $\epsilon$ smaller and find a larger n).

   (d) Well the typical set is
   $A_\epsilon^n = x^n : |\frac{1}{n} log(\frac{1}{p(x^n)} - H(X)| < \epsilon$
   We also know that for large enough n
   $Prob(X^n \epsilon A_\epsilon^n) > 1 - \epsilon$
   Thus

   i. first choose n to make $\epsilon$ tiny
   ii. then code $M = 2^{n(H(X)+\epsilon)}$ source sequences of length n which satisfy the requirements to be in $A_\epsilon^n$

   (e) Only change is that H(X) in $A_\epsilon^n$ definition becomes entropy rate $\mathcal{H}(X)$

3. (a) The tree diagram appears as shown in the figure above.
   So we want to force a merge in the last stage
   $P_4 + P_3 < P_1$ so that $P_2 + (P_3 + P_4)$ are merged rather than $P_3 + P_4$ and $P_1$
   ¿From statement we know
   $P_1 \geq P_2 \geq P_3 \geq P_4$
   Largest $P_4 + P_3$ can be is $P_2$ so set $= P_2$
   $2P_2 < P_1$ and $3P_2 + P_1 = 1 \to P_2 = \frac{1-P_1}{3}$
   $\to \frac{2(1-P_1)}{3} < P_1$
   $\to \frac{2}{3} < \frac{5}{3} P_1$
   $\to P_1 > \frac{2}{5} = .4$

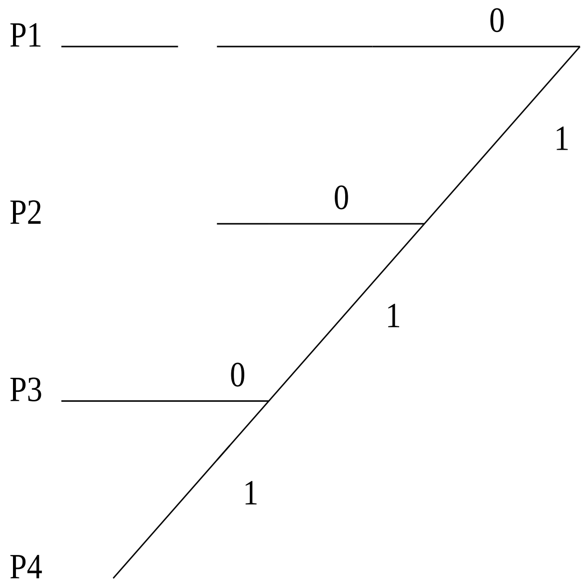


Figure 1: THIS IS FIG 3 (a)

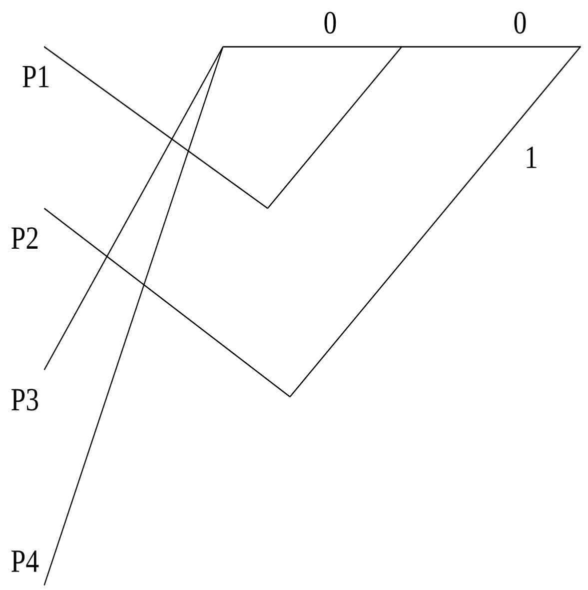


Figure 2: THIS IS FIG 3 (b)

(b) Now we want to force a merge early. The tree diagram appears as shown in the given figure.

$P_3 + P_4 > P_1 P_3 + P_4 + P_2 + P_1 = 1 \Rightarrow 2P_1 + P_2 < 1 P_2 \leq P_1 \Rightarrow 3P_1 < 1$

Therefore,

$P_1 < \frac{1}{3}$

4.

$$H(X/y) = -\sum_x p(x/y) log p(x/y)$$

Let p(x/0)= $\frac{1}{J}$ , x=0,1, ... J-1

p(x/1)=1, x=0

$$\begin{aligned} Letp(y) &= \frac{1}{M} \to y = 0 \\ &= 1 - \frac{1}{M} \to y = 1 \end{aligned}$$

$$p(x) = \sum_y p(x/y)p(y)$$

$$\begin{aligned} p(x) &= \frac{1}{MJ} + \frac{M-1}{M} \to x = 0 \\ &= \frac{1}{MJ} \to x = 1, 2, ...J-1 \end{aligned}$$

$H(X) = -(\frac{1}{MJ} + \frac{M-1}{M}) log(\frac{1}{MJ} + \frac{M-1}{M}) - \frac{J-1}{MJ} log(\frac{1}{MJ})$

Therefore,

$H(X) = (\frac{(M-1)J+1}{MJ} log(\frac{MJ}{(M-1)J+1} + \frac{J-1}{MJ} log(MJ)$

H(X/Y=0)=logJ

Let M be large, let $J >> M$

H(X) $\approx \frac{M-1}{M} log(\frac{M}{M-1}) + \frac{1}{M}$ logMJ

Therefore,

H(X) $\approx \frac{1}{M} log(MJ) = \frac{1}{M} log M + \frac{1}{M} log J < log J$ for $J >> M$

Therefore,

<u>H(X/y) is not $\leq$ H(X) for all y</u>

5. Theorem 5.4.3 states that a probability distribution $\underline{p}$ achieves capacity iff

$\sum_k Q_{k/j} log(\frac{Q_{k/j}}{q_k}) = C$ for $p_j \neq 0$

AND

$\sum_k Q_{k/j} log(\frac{Q_{k/j}}{q_k}) \leq C$ for $p_j = 0$

Let $Q^i_{k/j}$ be the transition prob matrix for the $i^{th}$ channel and $p_i(j)$ the distribution which achieves capacity.

Using the hint, we assign the new input distributions as

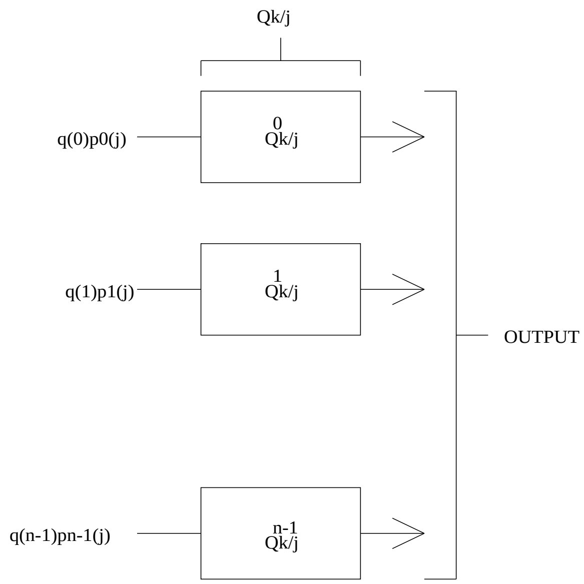


Figure 3: THIS IS FIG FOR P5.

$$\begin{bmatrix} q(0)p_0(j) \\ q(1)p_1(j) \\ . \\ . \\ q(n-1)p_{n-1}(j) \end{bmatrix}$$

Now calculate

$$\sum_k Q_{k/j} log(\frac{Q_{k/j}}{q_k})$$

for each channel

Channel 0 :

$$\sum_k Q^0_{k/j} log(\frac{Q^0_{k/j}}{\sum_j q(0)p_0(j)Q^0_{k/j}}) = C$$

.

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Channel n-1:

$$\sum_k Q^{n-1}_{k/j} log(\frac{Q^{n-1}_{k/j}}{\sum_j q(n-1)p_{n-1}(j)Q^{n-1}_{k/j}}) = C$$

For each Channel:

$$\sum_k Q^i_{k/j} log(\frac{Q^i_{k/j}}{\sum_j q(i)p_i(j)Q^i_{k/j}})$$

$$= -\sum_k Q^i_{k/j} log(q(x)) + C^i$$

$$= -log(q(i)) + C_i = C$$

Thus ( using $\log_2$ )

$q(i) = 2^{C^i - C}$

But $\sum q(x) = 1$

Therefore,

$$\sum_{i=0}^{n-1} q(i) = (\sum_{i=0}^{n-1} 2^{C^i})2^{-C} = 1$$

$$\Rightarrow C = \log_2 \sum_i 2^{C^i}$$

<u>IMPORTANT</u>

Note that the result is $\leq$ C for any $p_i(j) = 0$ since

$$\sum_k Q^i_{k/j} log(\frac{Q^i_{k/j}}{\sum p_i(j) Q_{k/j}}) \leq C^i$$

for any $p_i(j) = 0$