Rutgers University
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330:542 | Information Theory and Coding | Spring 1997

Problem Set 5 Solutions

1. C&T problem 9.1

   Differential Entropy

   (a) $f(x) = \lambda e^{-\lambda x}$ x ≥ 0

   $$h(f) = -\int_0^\infty \lambda e^{-\lambda x} log(\lambda e^{-\lambda x}) dx$$

   = -[E(log λ) - E(λx)] = -logλ + 1

   (b) $f(x) = \frac{\lambda}{2} e^{\lambda |x|}$

   $$h(f) = -\frac{1}{2}\int_0^\infty \lambda e^{-\lambda x} log(\frac{\lambda}{2} e^{-\lambda x}) dx - \frac{1}{2}\int_{-\infty}^\infty \lambda e^{\lambda x} log(\frac{\lambda}{2} e^{\lambda x}) dx$$

   $= \frac{1}{2}(-log\frac{\lambda}{2} + 1) + \frac{1}{2}(-log\frac{\lambda}{2} + 1) = -log\frac{\lambda}{2} + 1$

   (c) $X_1 \sim N(\mu_1, \sigma_1^2)$ $X_2 \sim N(\mu_2, \sigma_2^2)$ (independent) Y= $X_1 + X_2$ Therefore, $Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$
   $h(Y) = \frac{1}{2} log 2\pi e(\sigma_1^2 + \sigma_2^2)$

2. C&T problem 9.4

   Uniform Noise

   $$f(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & \text{o.w.} \end{cases}$$

   $$g(z) = \begin{cases} \frac{1}{a} & |z| \leq \frac{a}{2} \\ 0 & \text{o.w.} \end{cases}$$

   Y=X+Z $\Leftarrow$ channel and assume X,Z independent

   (a) $I(X:Y) = h(Y) - h(Y|X)$
   Y$\sim$f()$*$g()
   and $h(y|x) = h(z) = log(a)$
   Preliminaries
   Let

   $$T_\Delta(x) = \begin{cases} \frac{2x}{\Delta^2} & 0 \leq x \leq \Delta \\ 0 & \text{o.w.} \end{cases}$$

Triangular Distribution

$$
\begin{aligned}
h(T_\Delta) &= -\int_0^\Delta \frac{2x}{\Delta^2} log \frac{2x}{\Delta_2} dx \\
&= \frac{-2}{\Delta^2}\int_0^\Delta x log \frac{2x}{\Delta^2} dx - \frac{-2}{\Delta^2}\int_0^\Delta x log x dx \\
&= -log\frac{2}{\Delta^2} - \frac{2}{\Delta^2}\left(\frac{x^2}{2}log x - \frac{x^2}{4}\right)_0^\Delta \\
&= log\frac{\Delta^2}{2} - log\Delta + \frac{1}{2} \\
&= log\frac{\Delta}{2} + \frac{1}{2}
\end{aligned}
$$

Also $h(T_\Delta(-x)) = h(T_\Delta(x)) = h(T_\Delta(x-r))$ (r a constant)

<u>Now</u>

a$\leq$ 1

$$
f(y) = \begin{cases} \frac{1}{a}(y+\frac{a+1}{2}) & -(\frac{a-1}{2}) \leq y \leq -(\frac{1-a}{2}) \\ 1 & -(\frac{1-a}{2}) \leq y \leq (\frac{1-a}{2}) \\ \frac{1}{a}(\frac{a+1}{2}-y) & (\frac{1-a}{2}) \leq y \leq (\frac{a+1}{2}) \end{cases}
$$

and

a$\geq$1

$$
f(y) = \begin{cases} \frac{1}{a}(y+\frac{a+1}{2}) & -(\frac{a+1}{2}) \leq y \leq -(\frac{a-1}{2}) \\ \frac{1}{a} & -(\frac{a-1}{2}) \leq y \leq (\frac{a-1}{2}) \\ \frac{1}{a}(\frac{a+1}{2}-y) & (\frac{a-1}{2}) \leq y \leq (\frac{a+1}{2}) \end{cases}
$$

but

a$\leq$1

$f(y) = \frac{a}{2}T_a(y+\frac{a+1}{2}) + (1-a)U_{1-a}(y) + \frac{a}{2}T_a(\frac{a+1}{2}-y)$

$f(y) = P_1 f(y|E_1) + P_2 f(y|E_2) + P_3 f(y|E_3)$

where $E_1 \equiv y < -(\frac{1-a}{2})$

$E_2 \equiv |y| < \frac{1-a}{2}$

$E_3 \equiv y > \frac{1-a}{2}$

$h(f) = \frac{a}{2}h(T_a) + \frac{a}{2}h(T_a) + (1-a)h(U_{1-a}) + h(P_i)$

$= a(log\frac{a}{2}+\frac{1}{2}) + (1-a)log(1-a) - (1-a)log(1-a) - \frac{a}{2}log\frac{a}{2} - \frac{a}{2}log\frac{a}{2}$

$= \frac{a}{2} \Rightarrow a \leq 1$

I(X:Y)=$\frac{a}{2} - log a$

a$\geq$1

$f(y) = \frac{1}{2a}(T_1(y+\frac{a-1}{2}) + T_1(-y+\frac{a+1}{2})) + \frac{a-1}{a}U_{a-1}(y)$

<u>Same Reasoning</u>

$h(y) = \frac{-2}{2a}log\frac{1}{2a} - \frac{a-1}{a}log(\frac{a-1}{a}) + \frac{2}{2a}h(T_1) + \frac{a-1}{a}log(a-1)$

$= -\frac{1}{a}log\frac{1}{2a} - \frac{a-1}{a}log(\frac{a-1}{a}) + \frac{1}{a}(log\frac{1}{2}+\frac{1}{2}) + \frac{a-1}{a}log(a-1)$

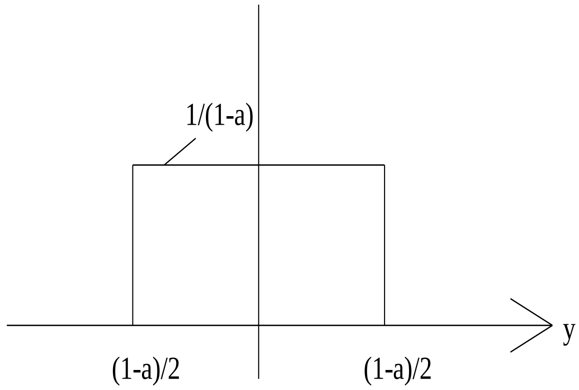

Figure 1: THIS IS FIG FOR P9.4

$= -\frac{1}{a}log\frac{1}{a} - \frac{a-1}{a}log\frac{1}{a} + \frac{1}{2a}$
$= \frac{1}{2a} + loga$
Therefore, $I(X:Y) = \frac{1}{2a}$ a≥ 1

(b) a=1

$$f(z) = \begin{cases} 1 & |z| \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

and
$|x| \leq \frac{1}{2}$
$C = max_{f(x)\epsilon F} I(X:Y)$
$F \equiv \{f(x)|f(x) = 0 \text{ if } |x| > \frac{1}{2}\}$
Slightly different than power constraint. Need to maximize h(y) subject to this constraint.

Y is limited to (-1,-1). Take a guess and make analogy between max entropy and disorder.
$U_2(x)$ gives maximum uncertainty on (-1,1)
<u>Prove it</u>
$D(f|U_2) \geq$0 fϵF
$-\int flog\frac{U_2}{f} = h(f) + \int flogU_2 = h(f) + log\frac{1}{2} = h(f) - log2$
Base 2
h(f)-1$\leq$0
$h(f) = 1$ iff $f = U_2$ and $f = U_2$ if
$f(x) = \frac{1}{2}\delta(x - \frac{1}{2}) + \frac{1}{2}\delta(x + \frac{1}{2})$
C=1 bit/channel use
<u>Could also do with variational Calculus</u>

3. C&T problem 10.2

   <u>2-Look Channel</u>

   $p(Y_1Y_2|X) = p(Y_1|X)p(Y_2|X)$

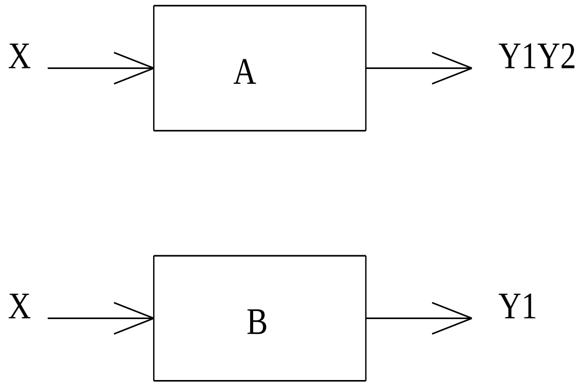

Figure 2: THIS IS FIG FOR P10.2

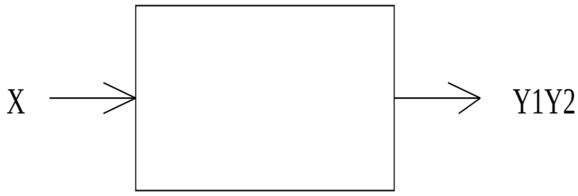

Figure 3: THIS IS FIG FOR P10.3

(a)

$$\begin{aligned}
I(X;Y_1,Y_2) &= h(Y_1Y_2) - h(Y_1Y_2|X) \\
&= h(Y_1Y_2) - h(Y_1|X) - h(Y_2|X) \\
&= h(Y_1Y_2) + 2h(Y_1) - 2h(Y_1) - 2h(Y_2|X) + h(Y_1|X) - h(Y_2|X) \\
&= 2I(X;Y_1) + h(Y_1Y_2) - 2h(Y_1) + h(Y_1|X) - h(Y_2|X)(cond.i.i.d) \\
&= 2I(X;Y_1) + h(Y_1Y_2) - h(Y_1) - h(Y_2) \\
&= 2I(X;Y_1) - D(Y_1||Y_2) = 2I(X;Y_1) - I(Y_1;Y_2)
\end{aligned}$$

(b) Please see the figure given
$C_A = maxI(X;Y_1,Y_2) \leq 2maxI(X;Y1) \leq 2C_B$
Obviously holds for discrete channels too

4. C&T problem 10.3

<u>2-Look Gauss</u>

Please see the figure given above

$Y_1 = X + Z_1$

$Y_2 = X + Z_2$

$$K_Z = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} N_0$$

$|\rho| \leq 1$

$E(X^2) \leq \rho$

$\underline{z} \sim N(0, K_Z)$

Lets do this in general $I(X;Y_1Y_2) = h(Y_1Y_2) - h(Y_1Y_2|X)$

$h(Y_1Y_2|X) = \frac{1}{2}log(2\pi e)^2|K_Z| = \frac{log(2\pi e)^2(1-\rho^2)N_0^2}{2}$

(Y_1Y_2 jointly Gaussian given X) $p(\underline{y}|x) = N((x,x), K_Z)$

$I(XY_1Y_2) = h(Y_1Y_2) - \frac{1}{2}log(2\pi e)^2(1+\rho)(1-\rho)$

Need max $h(Y_1Y_2) \Rightarrow$ Max power

$Y_1 \Rightarrow N_0 + P$

$Y_2 \Rightarrow N_0 + P$

$E(Y_1Y_2) = E(X^2) + \rho N_0$

$$K_Y = \begin{bmatrix} N_0 + P & \mathrm{P}+\rho N_0 \\ P + \rho N_0 & N_0+\mathrm{P} \end{bmatrix}$$

Entropy max if Y_i are jointly gaussian so

$X \sim N(0, \rho)\underline{Y} \sim N(0, K_Y)$

$$\begin{aligned} C &= \tfrac{1}{2}log(2\pi e)^2((N_0+P)^2 - (P+\rho N_0)^2) - \tfrac{1}{2}log(2\pi e)^2(1-\rho)(1+\rho)N_0^2 \\ &= \tfrac{1}{2}log(\tfrac{N_0^2+P^2+2N_0P-P^2-\rho^2N_0^2-2\rho N_0P}{(1-\rho)(1+\rho)N_0^2}) \\ &= \tfrac{1}{2}log(\tfrac{N_0(N_0+2P-\rho^2N_0-2\rho P}{(1-\rho)(1+\rho)N_0^2}) \\ &= \tfrac{1}{2}log(\tfrac{N_0^2(1-\rho^2)+2PN_0(1-\rho)}{(1-\rho)(1+\rho)N_0^2}) \end{aligned}$$

$C = \frac{1}{2}log(1 + \frac{2P}{(1+\rho)N_0})$

(a) p=1 $\Rightarrow z_1 = z_2$ so no new info added by second look
    $C = \frac{1}{2}log(1 + \frac{P}{N_0})$

(b) p=0
    no correlation in z_i. Could average $Y_1 + Y_2$ to obtain $Y_3 = X + \frac{Z_1+Z_2}{2} \Rightarrow$ power $\frac{N_0}{2}$
    $C = \frac{1}{2}log(1 + \frac{2P}{N_0})$

(c) p=-1
    Qualitatively this should be so since $Y_1 + Y_2 = X$
    $C = \frac{1}{2}log(1 + \frac{2P}{0.N_0}) = \infty$

5. C&T problem 10.4

   Parallel Channels and Water Filling

   Please see the given figure.

   for $2P > \sigma_1^2 - \sigma_2^2$ we begin to use channel 1.

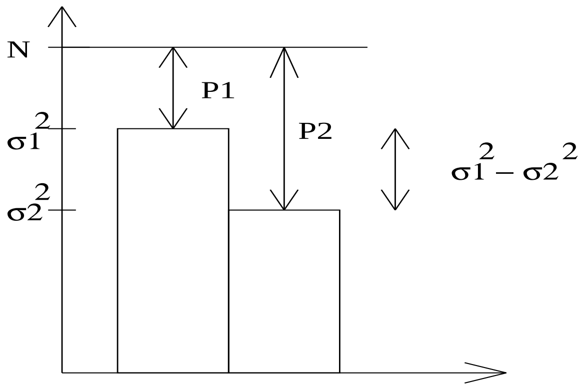

Figure 4: THIS IS FIG FOR P10.4