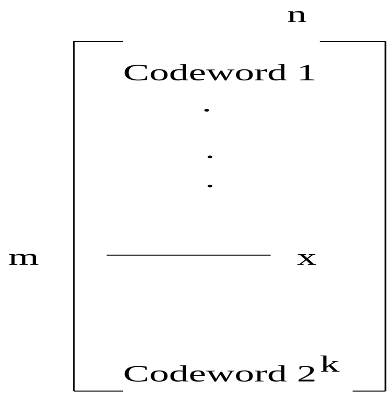

Figure 1: MATRIX 1

Rutgers University
The State University of New Jersey
College of Engineering
Department of Electrical and Computer Engineering

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Problem Set 6 Solutions

1. Fun with Parity Check codes

   (a) Code linearity is the key
   $X_1 + X_2 \epsilon$C
   0 ϵ C
   Since 0 ϵ C, codewords can't all be odd
   even + even = even $\Rightarrow$ could be all even
   odd + odd = even
   odd + even = odd
   Set of even codewords $E = \{0, e_1, e_2, ..., e_k\}$
   odd $O = \{\sigma_0, ..., \sigma_l\}$
   So,
   $\sigma_0 + e_i = \sigma_{r(i)}$ for some r and $\sigma_{r(i)} \neq \sigma_{r(j)}$ i$\neq$j
   Therefore, each odd codeword is generated by pairing σ_0 with one of k+1 even codewords
   $\Rightarrow$ l=k
   n=2(k+1)
   Therefore,
   half even/half odd

   (b) (See matrix 1)
   Contention is that each column contains either all zeros or half+half zeros+ 1's
   Same idea as in (a). Put all codewords with 1 in n^{th} position on top
   (See matrix 2)
   $e_i + e_j = e_k$
   $\sigma_i + \sigma_j = e_l$

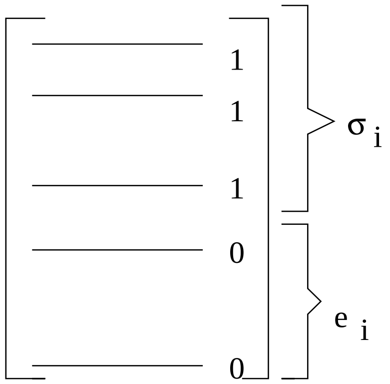

Figure 2: MATRIX 2

$e_i + \sigma_0 = \sigma_{j(i)}$ and $\sigma_{j(i)} \neq \sigma_{j(m)}$ i$\neq$m
So every σ_i codeword corresponds to a unique e_i
Therefore, half 1's half zeros or all zero (no σ_i)
Obviously should drop n^{th} digit if it's always zero since it conveys no info

(c) Each column has at most half zeros

$$\overline{N} = \frac{1}{2^k}\sum_{i=1}^{2k}\sum_{j=1}^{n} C_{ij} = \frac{1}{2^k}\sum_{j=1}^{n}\sum_{i=1}^{2k} C_{ij} \leq \frac{1}{2^k}\sum_{j=1}^{n}\sum_{i=1}^{2k}\frac{n}{2} = \frac{n}{2}$$

2. <u>Plotkin Bound</u>

(a) We know E(1's)$\leq \frac{n}{2}$ (N,L) code $\rightarrow 2^L$ codewords
Conditioning on $\overline{0}$ (not zero),
$E(w(C_i)) = E(1's) \leq \frac{N}{2}\frac{2^L}{2^L-1}$

$$d_{min} = \min_i w(C_i) = \min_i ones(C_i)$$

$w/C_i \neq$0
where ones() is the # of ones in the argument
$d_{min} \leq E(ones(C_i)) \leq \frac{N}{2}\frac{2^L}{2^L-1}$
because smallest $\leq$ mean
(Yes it is obvious but region is important)

(b) Doing the bonus does it all, so lets do bonus!
uG=C
u $\rightarrow$ L bits
C $\rightarrow$ N bits
Choose all codewords with zero in digit i. Assuming the code is irreducible, this new set of codewords has cardinality 2^{L-1} (part (b) of previous problem)
Therefore, $d'_{min} \leq \frac{N-1}{2}\frac{2^{L-1}}{2^{L-1}-1}$
Choose 2 zero digits in succession
$d''_{min} \leq \frac{N-2}{2}\frac{2^{L-2}}{2^{L-2}-1}$
Iterate down to $\frac{N-L+1}{2}\frac{2'}{2-1} \Leftarrow$ two codewords

$$
\begin{array}{cc}
 & \begin{array}{cc} I_3 & P \end{array} \\
G_1 = & \begin{bmatrix} 1001101 \\ 0101011 \\ 0010111 \end{bmatrix}
\end{array}
$$

(a)

Figure 3: MATRIX 3

Since these are all part of the same code
$d_{min} \le \frac{N-L+j}{2}\frac{2^j}{2^j-1}$ j=1,2...L

(c) N$\ge$ 2 d_{min}-1
Previous result 2 $d_{min} \le (N-L+j)(\frac{2^j}{2^j-1})$
$(2d_{min})(\frac{2^j-1}{2^j}) - j \le N-L$
Let j=1+$\lfloor log_2 d_{min} \rfloor$
$N-L \ge (2d_{min})(\frac{2.2^{\lfloor log_2 d_{min} \rfloor}-1}{2.2^{\lfloor log_2 d_{min} \rfloor}}) - (1+\lfloor log_2 d_{min} \rfloor)$
$= 2d_{min}(1-\frac{1}{2}2^{-\lfloor log_2 d_{min} \rfloor}) - 1 - \lfloor log_2 d_{min} \rfloor$
$= 2(d_{min} - 2^{log d_{min} - \lfloor log_2 d_{min} \rfloor}) - 1 - \lfloor log_2 d_{min} \rfloor$
(We know that $2^{log d_{min} - \lfloor log_2 d_{min} \rfloor} < 1$)
$\Rightarrow > 2d_{min} - 3 - \lfloor log_2 d_{min} \rfloor$
Therefore, N-L$\ge 2(d_{min}-1) - \lfloor log_2 d_{min} \rfloor$
because N-L is an integer

(d) $\frac{L}{N} \le \frac{-2(d_{min}-1)+\lfloor log_2 d_{min} \rfloor}{N} + 1$
$= 1 - \frac{2d_{min}}{N} + \frac{2+\lfloor log_2 d_{min} \rfloor}{N}$
with $\frac{d_{min}}{N} < \frac{1}{2} \Rightarrow N > 2d_{min}$
So,

$$\lim_{N\to\infty} \frac{L}{N} \le 1 - \frac{2d_{min}}{N} + \frac{2}{N} + \frac{\lfloor log_2 \frac{N}{2} \rfloor}{N} = 1 - \frac{2d_{min}}{N}$$

3. <u>Parity Check codes</u>

(See Matrix 3)
looks hamming! (but this is the generator not the PC matrix)
(See matrices 4 and 5)

(b) 16 possible syndromes $\Rightarrow$ choose maximum weight error sequence decoding
7 single digits $\Rightarrow$

$$\begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

$= 21$ 2 digits $\Rightarrow$ errors
(See figure 6)

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \\ P \\ \\ \\ \\ I_4 \\ \\ \end{array}$$

Figure 4: MATRIX 4

$$H_{II} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 5: MATRIX 5

	errors	syndrome		
1	0 0 0 0 0 0 0	0 0 0 0	0	
2	1 0 0 0 0 0 0	1 1 0 1	13	
3	0 1 0 0 0 0 0	1 0 1 1	11	
4	0 0 1 0 0 0 0	0 1 0 1	5	
5	0 0 0 1 0 0 0	1 0 0 0	8	7 1-bit errors
6	0 0 0 0 1 0 0	0 1 0 0	4	
7	0 0 0 0 0 1 0	0 0 1 0	2	
8	0 0 0 0 0 0 1	0 0 0 1	1	
9	1 1 0 0 0 0 0	0 1 1 0	6	
	1 0 1 0 0 0 0	no good		
	1 0 0 1 0 0 0	no good		
10	1 0 0 0 1 0 0	1 0 0 1	9	
11	1 0 0 0 0 1 0	1 1 1 1	15	8 2-bit errors
12	1 0 0 0 0 0 1	1 1 0 0	12	
13	0 1 1 0 0 0 0	1 1 1 0	14	
14	0 1 0 1 0 0 0	0 0 1 1	3	
15	0 0 1 0 0 1 0	0 1 1 1	7	
16	0 0 0 1 0 1 0	1 0 1 0	10	

Figure 6: THIS IS FIG FOR PARITY CHECK CODES

	error sequence	syndrome	
0	0 0 0 0 0 0 0	0 0 0 0	0
1	0 0 0 0 0 0 1	0 0 0 1	1
2	0 0 0 0 0 1 0	0 0 1 0	2
3	0 0 0 0 1 0 0	0 1 0 0	4
4	0 0 0 1 0 0 0	1 0 0 0	8
5	0 0 1 0 0 0 0	0 1 1 1	7
6	0 1 0 0 0 0 0	1 0 1 1	11
7	1 0 0 0 0 0 0	1 1 0 1	13
8	0 0 0 0 0 1 1	0 0 1 1	3
9	0 0 0 0 1 0 1	0 1 0 1	5
10	0 0 0 1 0 0 1	1 0 0 1	9
11	0 0 1 0 0 0 1	0 1 1 0	6
12	0 1 0 0 0 0 1	1 0 1 0	10
13	1 0 0 0 0 0 1	1 1 0 0	12
14	0 0 0 0 1 1 0	0 1 1 0	can't use this one
15	0 0 0 1 0 1 0	1 0 1 0	can't use this one
		need 14,15 $\Rightarrow$ 1110,1111	
14	0 0 0 1 1 1 0	1 1 1 0	can only do with 3 errors
15	0 0 1 1 0 0 0	1 1 1 1	

Figure 7: THIS IS FIG FOR PARITY CHECK CODES

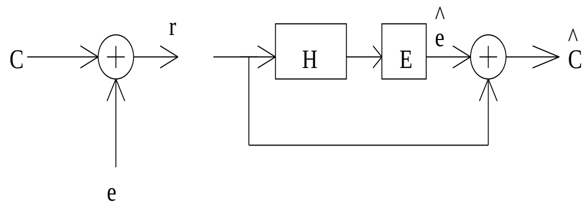

Figure 8: THIS IS FIG FOR PARITY CHECK CODES

(c) $P_{II} = 1 - (1-\epsilon)^7 - 7\epsilon(1-\epsilon)^6 - 8\epsilon^2(1-\epsilon)^5$
$P_I > P_{II}$
(See figure 7)
Since code is systematic we check syndrome and look at first 3 digits
(See figure 8)

(d) (See figure 9)

(e) Too simple (you already did it)

4. <u>Parity Check codes with same block length</u>

$xC_1 \epsilon C_1$ $uC_2 \epsilon C_2$ over all L-tuples x,u

Let $C^* = \{c | c\epsilon C_1 \cap C_2\}$

Let $x_1 C_1 = C_1^* = u_1 C_2$

$x_2 C_1 = C_2^* = u_2 C_2 \Rightarrow C_1^* + C_2^* \epsilon C^*$

So C^* is a linear subspace of C_1 and $C_2 \Rightarrow$ new P.C.code

$C_i^* H_1 = C_i^* H_2 = 0$

so $C_i^* \perp$ to columns of H_1 and H_2

So find basis for space spanned by these columns (concatenate columns and <u>remove dependent ones</u>)

codewords for I	code II
0000000	0000000
0010111	0010101
0101011	0101011
0111100	0111110
1011010	1001101
1100110	1011000
1110001	1100110
1001101	1110011

$d_{min}=4$ (codewords for I)

$d_{min}=3$ (code II)

Figure 9: THIS IS FIG FOR PARITY CHECK CODES

5. Add a PC digit

   Can only increase w_{min} so

   w_{min} odd $\rightarrow$ add up digits $\rightarrow$ 1 append $\rightarrow$ $w'_{min} = w_{min} + 1$

   w_{min} even $\rightarrow$ add up digits $\rightarrow$ 0 append $\rightarrow$ w_{min} unchanged