State Estimation, Wireless Tropes, Demons and Uncertainty

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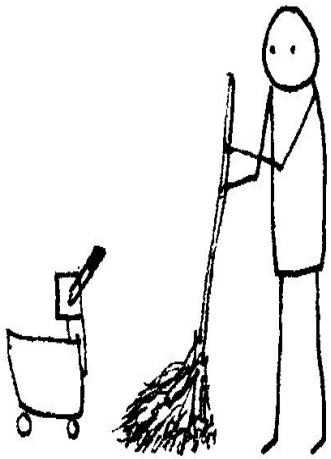
PHYSICIST

Communication Theorist



But EVERYTHING Communicates

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Atoms within objects

The walls and the air

The birds and the bees

Cells, Organisms & Ecologies

Economies & Societies

The Universe

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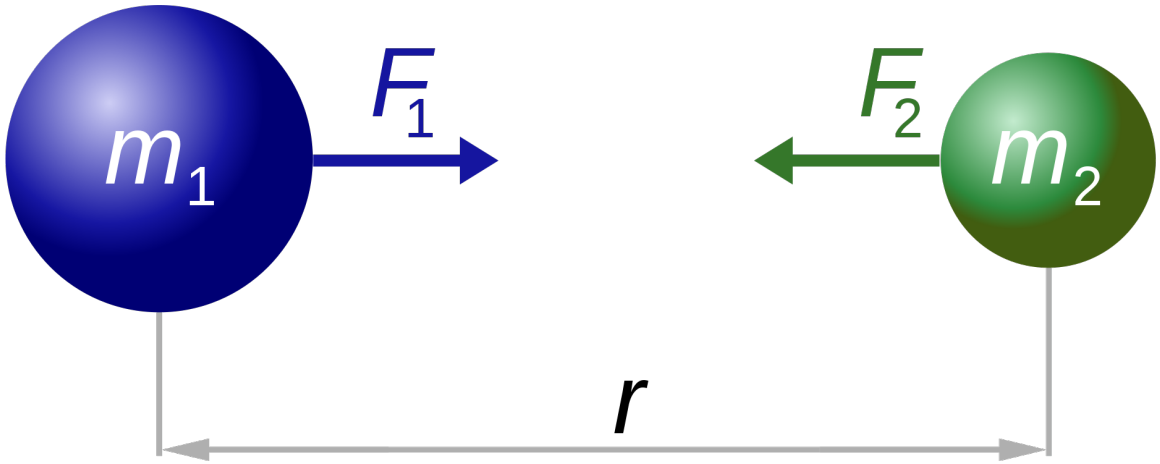
What are they saying and can we listen in?

A Deceptively Simple Problem



$$F_1 = F_2 = G\frac{m_1 \times m_2}{r^2}$$

Dynamics

Dynamics

$$m_1\ddot{x}_1 = \frac{m_1 m_2 G}{(x_1 - x_2)^2}\left(\frac{x_2 - x_1}{|x_2 - x_1|}\right)$$

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Coupled Differential Equations

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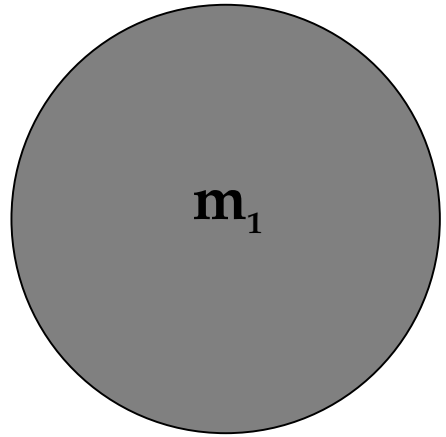
Coupled Differential Equations

DONE! Right????

It's Lonely Out in Space



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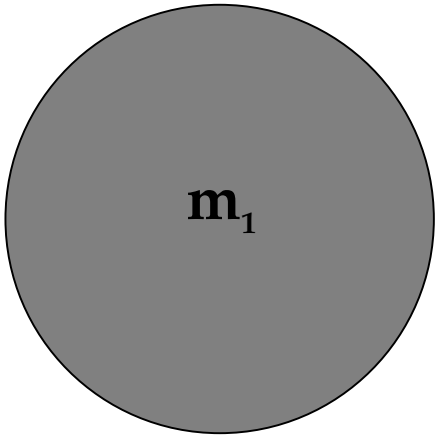
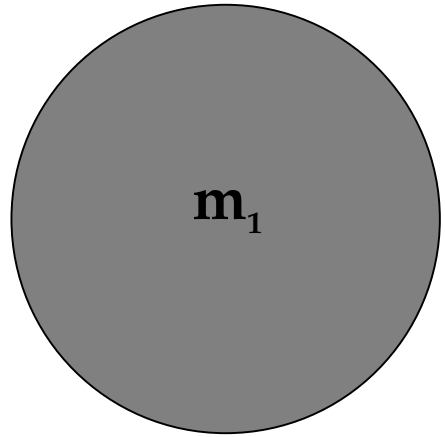


How does m_1 know m_2 and $x_1 - x_2$?

It's Lonely Out in Space



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Through what mechanism?

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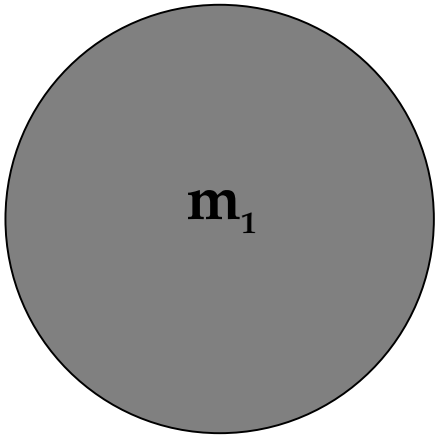

How does m_1 know m_2 and $x_1 - x_2$?

Through what mechanism?

With what precision?

Bottom Line

m_1 and m_2 are "talking"

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Inifnite Precision? $\rightarrow$ Ridiculous!

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$$F = \frac{m_1 m_2 G}{R^2}$$

Unfortunately

Unfortunately

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I

Have

Unfortunately

I
Have
No

Unfortunately

I
Have
No
Real

Unfortunately

I
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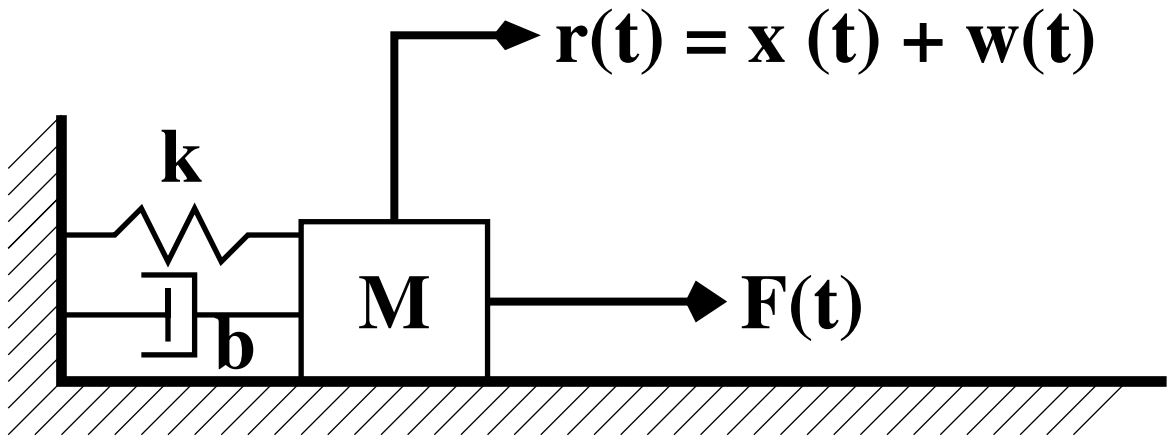
No
Real
Answers

What We're Actually Gonna Analyze: watching vs. telling



$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}F(t)$$
$$r(t) = \mathbf{C}\mathbf{x}(t) + w(t)$$
$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

Undriven Homogeneous Case

Linear systems 101: linearly independent functions $\{\psi_n()\}$

$$x(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + w(t) = \sum_{n=1}^{N} a_n(\mathbf{x}_0)\psi_n(t) + w(t)$$

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A dash of signal space: Derive orthonormal basis $\{\phi_n() \}$ on $[0, T]$:

$$x(t) = \sum_{n=1}^{N} b_n(\mathbf{x}_0)\phi_n(t) + \sum_{n=1}^{N} w_n\phi_n(t)$$

Projection

Project $r(t)$ onto basis functions:

$$r_n = \langle r(t), \phi_n(t) \rangle$$

Define

$$\mathbf{G}_n = \left\langle e^{\mathbf{A}t}, \phi_n(t) \right\rangle$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{C}\mathbf{G}_1 \\ \mathbf{C}\mathbf{G}_2 \\ \vdots \\ \mathbf{C}\mathbf{G}_N \end{bmatrix}$$

Wirless Tropes: a cacophony of voices

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Colored Noise Channel

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Multiple Access Channel

$$\mathbf{r} = \sum_{n=1}^{N} \mathbf{q}_n (\mathbf{x}_0)_n + \mathbf{w}$$

($\mathbf{q}_n \leftrightarrow n^{th}$ column of $\mathbf{Q}$)

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energy constraints on system elements $\Leftrightarrow \mathbf{x}^\top \mathbf{Z} \mathbf{x} \leq \mathcal{E}$

WATCHING: mmse estimation of $\mathbf{x}(0)$

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- Assume: $\mathbf{x}(0) \sim \mathcal{N}(\mathbf{0}, \mathbf{K}_{\mathbf{x}(0)})$
- Energy Constraint: Trace$[\mathbf{K}_{\mathbf{x}(0)}] = \mathcal{E}$
- Linear Estimator Best:

$$\hat{\mathbf{x}}(0) = \mathbf{K}_{\{\mathbf{x}(0),\mathbf{r}\}}\mathbf{K}_{\mathbf{r}}^{-1}\mathbf{r}$$

- Error Covariance:

$$\mathbf{K}_e = \mathbf{K}_{\mathbf{x}_0} - \mathbf{K}_{\mathbf{x}(0)}\mathbf{Q}^\top\mathbf{K}_{\mathbf{r}}^{-1}\mathbf{Q}\mathbf{K}_{\mathbf{x}(0)}$$

- Total MMSE Error: Trace$[\mathbf{K}_e]$

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- Energy Constraint: $E\left[|\mathbf{x}_0[\ell]|^2\right] = \mathcal{E}/L$
- Receiver sees: $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_L$

Per-$\mathbf{r}_\ell$ Channel Capacity:

$$C_{\text{color}} = \max_{\text{Trace}[\mathbf{K}_{\mathbf{x}_0}]=\frac{\mathcal{E}}{L}} \frac{1}{2}\left(\log|\mathbf{K}_{\mathbf{x}_0} + \mathbf{K}_{\tilde{\mathbf{w}}}| - \log|\mathbf{K}_{\tilde{\mathbf{w}}}|\right)$$

where $\tilde{\mathbf{w}} = \mathbf{Q}^{-1}\mathbf{w}$

Multiple Independent Demons Send Coded State: MAC

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Use Multiuser Detection

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Same as Colored Noise

Multiple Independent Demons Send Coded State: MAC

Use Multiuser Detection

Same as Colored Noise

Will Ignore Here

Taking $L \to \infty$

- $\mathbf{K_w} = \mathcal{N}_0 \mathbf{I}$
- $\mathbf{K_{\tilde{w}}} = \mathcal{N}_0 \mathbf{Q}^{-1} (\mathbf{Q}^{-1})^\top$
- λ_{min}: smallest eigenvalue of $\mathbf{Q}^{-1} (\mathbf{Q}^{-1})^\top$
- **Capacity Approximation:**

$$C_{\text{color}} \approx \frac{1}{2} \log \left(\frac{\mathcal{E}/\mathcal{N}_0}{L \lambda_{\text{min}}} + 1 \right)$$

- **Bits Conveyed:**

$$B_{\text{color}} = L C_{\text{color}} \approx \frac{1}{2} \frac{\mathcal{E}}{\mathcal{N}_0 \lambda_{\text{min}}}$$

Rate Distortion (quantizing $\mathbf{x}(0)$)

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- N i.i.d. Gaussians with a total of B nats $\rightarrow$ total distortion:

$$\mathcal{D} = N\sigma^2 e^{-2B/N}$$

Comparisons ($\mathbf{K}_{\mathbf{x}(0)} = \mathcal{E}\mathbf{I}/2$, $\eta = \mathcal{E}/\mathcal{N}_0$)

Overdamped:

	$\eta = 0.1$	$\eta = 1$	$\eta = 10$
MMSE	0.97	0.81	0.56
Color Demon	0.93	0.53	1.9×10^{-3}

Critically Damped:

	$\eta = 0.1$	$\eta = 1$	$\eta = 10$
MMSE	0.96	0.78	0.47
Color Demon	0.93	0.48	6.4×10^{-4}

Underdamped:

	$\eta = 0.1$	$\eta = 1$	$\eta = 10$
MMSE	0.98	0.89	0.57
Color Demon	0.96	0.77	6.9×10^{-2}

Uncertainty Principle (quantum mechanics)

- σ_x^2: position uncertainty variance
- σ_p^2: momentum uncertainty variance
- $\hbar$: Planck's constant

$$\boldsymbol{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$$

Bit Budgeting

- x **and** $\dot{x}$**:** state variables
- B_x: nats to specify x
- $B_{\dot{x}}$: nats to specify $\dot{x}$
- B_{color}: nat budget

$$B_x + B_{\dot{x}} \leq B_{\text{color}}$$

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$$\boldsymbol{\sigma_x \sigma_p = \frac{\sigma^2}{M} e^{-(B_x + B_{\dot{x}})} \geq \frac{\sigma^2}{M} e^{-B_{\text{color}}} = \frac{\sigma^2}{M} e^{-\frac{\eta}{2\lambda_{\text{min}}}}}$$

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- **NOT** claiming *equivalence* with Uncertainty Principle (yet 🙂)

Punchline

Punchline

All

Punchline

All
Physical Interaction

Punchline

All
Physical Interaction
Is

Punchline

All
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Communication

Punchline

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SO,

Communication/Information Theorists RULE!