State Estimation, Wireless Tropes, Demons and Uncertainty

Christopher Rose
Brown University, School of Engineering

ISIT ’16
Barcelona
July 14th, 2016
PHYSICIST
Communication Theorist
But EVERYTHING Communicates
But EVERYTHING Communicates

Atoms within objects
The walls and the air
The birds and the bees
Cells, Organisms & Ecologies
Economies & Societies
The Universe
But EVERYTHING Communicates

Atoms within objects
The walls and the air
The birds and the bees
Cells, Organisms & Ecologies
Economies & Societies
The Universe

What are they saying and can we listen in?
A Deceptively Simple Problem

\[ F_1 = F_2 = G \frac{m_1 \times m_2}{r^2} \]
Dynamics
Dynamics

\[ m_1 \ddot{x}_1 = \frac{m_1 m_2 G}{(x_1 - x_2)^2} \frac{x_2 - x_1}{|x_2 - x_1|} \]

\[ m_2 \ddot{x}_2 = -\frac{m_1 m_2 G}{(x_1 - x_2)^2} \frac{x_2 - x_1}{|x_2 - x_1|} \]
Wireless Tropes, Demons, Uncertainty

Newton Redux

Dynamics

\[ m_1 \ddot{x}_1 = \frac{m_1 m_2 G}{(x_1 - x_2)^2} \left( \frac{x_2 - x_1}{|x_2 - x_1|} \right) \]

\[ m_2 \ddot{x}_2 = -\frac{m_1 m_2 G}{(x_1 - x_2)^2} \left( \frac{x_2 - x_1}{|x_2 - x_1|} \right) \]

Coupled Differential Equations
Dynamics

\[ m_1 \ddot{x}_1 = \frac{m_1 m_2 G}{(x_1 - x_2)^2} \left( \frac{x_2 - x_1}{|x_2 - x_1|} \right) \]

\[ m_2 \ddot{x}_2 = -\frac{m_1 m_2 G}{(x_1 - x_2)^2} \left( \frac{x_2 - x_1}{|x_2 - x_1|} \right) \]

Coupled Differential Equations

DONE! Right????
It’s Lonely Out in Space
It’s Lonely Out in Space

How does $m_1$ know $m_2$ and $x_1 - x_2$?
It’s Lonely Out in Space

How does $m_1$ know $m_2$ and $x_1 - x_2$? Through what mechanism?
It’s Lonely Out in Space

How does $m_1$ know $m_2$ and $x_1 - x_2$?
Through what mechanism?
With what precision?
Bottom Line

$m_1$ and $m_2$ are “talking”
$m_1$ and $m_2$ are “talking”

How?
$m_1$ and $m_2$ are “talking”

How?

Infinite Precision? → Ridiculous!
Bottom Line

$m_1$ and $m_2$ are “talking”

How?

Infinite Precision? $\rightarrow$ Ridiculous!

$$F = \frac{m_1 m_2 G}{R^2}$$
Unfortunately
Unfortunately
Unfortunately

I

Have
Unfortunately

I

Have

No
Unfortunately I Have No Real
Unfortunately

I

Have

No

Real

Answers
What We’re Actually Gonna Analyze: watching vs. telling

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B F(t) \\
r(t) &= C x(t) + w(t) \\
x &= \begin{bmatrix} x \\ \dot{x} \end{bmatrix}
\end{align*}
\]
Undriven Homogeneous Case

**Linear systems 101:** linearly independent functions \( \{ \psi_n() \} \)

\[
x(t) = C e^{At} x(0) + w(t) = \sum_{n=1}^{N} a_n(x_0) \psi_n(t) + w(t)
\]

(\( \psi_n(t) \) could be \( e^{\lambda_n t} \) for unique evals of \( A \)).
Undriven Homogeneous Case

Linear systems 101: linearly independent functions \( \{\psi_n()\} \)

\[
x(t) = C e^{At} x(0) + w(t) = \sum_{n=1}^{N} a_n(x_0) \psi_n(t) + w(t)
\]

\( \psi_n(t) \) could be \( e^{\lambda_n t} \) for unique evals of \( A \).

A dash of signal space: Derive orthonormal basis \( \{\phi_n()\} \) on \([0, T]\):

\[
x(t) = \sum_{n=1}^{N} b_n(x_0) \phi_n(t) + \sum_{n=1}^{N} w_n \phi_n(t)
\]
Projection

Project $r(t)$ onto basis functions:

$$r_n = \langle r(t), \phi_n(t) \rangle$$

Define

$$G_n = \langle e^{At}, \phi_n(t) \rangle$$

$$Q = \begin{bmatrix} CG_1 \\ CG_2 \\ \vdots \\ CG_N \end{bmatrix}$$
Wirless Tropes: a cacophony of voices
**Wireless Tropes: a cacophony of voices**

**Colored Noise Channel**

\[ r = Qx_0 + w \]
Wireless Tropes: a cacophony of voices

Colored Noise Channel

\[ r = Qx_0 + w \]

Multiple Access Channel

\[ r = \sum_{n=1}^{N} q_n (x_0)_n + w \]

\( (q_n \leftrightarrow n^{th} \text{ column of } Q) \)
Wireless Tropes: a cacophony of voices

Colored Noise Channel

\[ r = Qx_0 + w \]

Multiple Access Channel

\[ r = \sum_{n=1}^{N} q_n(x_0)_n + w \]

\((q_n \leftrightarrow n^{th} \text{ column of } Q)\)

energy constraints on system elements \(\Leftrightarrow x^\top Zx \leq \mathcal{E}\)
WATCHING: mmse estimation of $x(0)$
WATCHING: mmse estimation of $x(0)$

- Assume: $x(0) \sim \mathcal{N}(0, \mathbf{K}_{x(0)})$

- Energy Constraint: $\text{Trace}[\mathbf{K}_{x(0)}] = \mathcal{E}$

- Linear Estimator Best:
  \[
  \hat{x}(0) = \mathbf{K}_{\{x(0), r\}} \mathbf{K}_r^{-1} r
  \]

- Error Covariance:
  \[
  \mathbf{K}_e = \mathbf{K}_{x0} - \mathbf{K}_{x(0)} \mathbf{Q}^\top \mathbf{K}_r^{-1} \mathbf{Q} \mathbf{K}_{x(0)}
  \]

- Total MMSE Error: $\text{Trace}[\mathbf{K}_e]$
TELLING: demon sends coded state (colored noise)
TELLING: demon sends coded state (colored noise)

- Successive $[0, T]$ intervals
**TELLING:** demon sends coded state (colored noise)

- Successive $[0, T]$ intervals

- Demon sets system state: $x_0[1], x_0[2], \ldots x_0[L]$
TELLING: demon sends coded state (colored noise)

- Successive $[0, T]$ intervals

- Demon sets system state: $x_0[1], x_0[2], \cdots x_0[L]

- Energy Constraint: $E \left[ |x_0[\ell]|^2 \right] = \mathcal{E} / L$
TELLING: demon sends coded state (colored noise)

- Successive \([0, T]\) intervals
- Demon sets system state: \(x_0[1], x_0[2], \cdots x_0[L]\)
- Energy Constraint: \(E \left[ |x_0[\ell]|^2 \right] = E / L\)
- Receiver sees: \(r_1, r_2, \cdots, r_L\)

**Per-\(r_\ell\) Channel Capacity:**

\[
C_{\text{color}} = \max_{\text{Trace}[K_{x_0}] = \frac{E}{L}} \frac{1}{2} \left( \log |K_{x_0} + K_{\tilde{w}}| - \log |K_{\tilde{w}}| \right)
\]

where \(\tilde{w} = Q^{-1}w\)
Multiple Independent Demons Send Coded State: MAC
Multiple Independent Demons Send Coded State: MAC

Use Multiuser Detection
Multiple Independent Demons Send Coded State: MAC

Use Multiuser Detection

Same as Colored Noise
Multiple Independent Demons Send Coded State: MAC

Use Multiuser Detection

Same as Colored Noise

Will Ignore Here
Taking $L \to \infty$

- $K_w = N_0 I$
- $K_{\tilde{w}} = N_0 Q^{-1}(Q^{-1})^T$
- $\lambda_{\text{min}}$: smallest eigenvalue of $Q^{-1}(Q^{-1})^T$

- **Capacity Approximation:**
  
  $$C_{\text{color}} \approx \frac{1}{2} \log \left( \frac{\mathcal{E}/N_0}{L\lambda_{\text{min}}} + 1 \right)$$

- **Bits Conveyed:**
  
  $$B_{\text{color}} = LC_{\text{color}} \approx \frac{1}{2} \frac{\mathcal{E}}{N_0 \lambda_{\text{min}}}$$
Rate Distortion (quantizing $x(0)$)
Rate Distortion (quantizing $x(0)$)

- $B$ nats to spend
Rate Distortion (quantizing $x(0)$)

- $B$ nats to spend
- Distortion $\mathcal{D}$
Rate Distortion (quantizing $x(0)$)

- $B$ nats to spend
- Distortion $\mathcal{D}$
- $X \sim \mathcal{N}(0, \sigma^2)$ and a square error distortion measure
Rate Distortion (quantizing $x(0)$)

- $B$ nats to spend
- Distortion $\mathcal{D}$
- $X \sim \mathcal{N}(0, \sigma^2)$ and a square error distortion measure
- Minimum distortion:
  \[ \mathcal{D} = \sigma^2 e^{-2B} \]
Rate Distortion (quantizing $x(0)$)

- $B$ nats to spend
- Distortion $\mathcal{D}$
- $X \sim \mathcal{N}(0, \sigma^2)$ and a square error distortion measure
- Minimum distortion:
  $$\mathcal{D} = \sigma^2 e^{-2B}$$
- $N$ i.i.d. Gaussians with a total of $B$ nats $\rightarrow$ total distortion:
  $$\mathcal{D} = N\sigma^2 e^{-2B/N}$$
**Comparisons** \((K_{x(0)} = \mathcal{E}I/2, \eta = \mathcal{E}/N_0)\)

**Overdamped:**

<table>
<thead>
<tr>
<th>(\eta = 0.1)</th>
<th>(\eta = 1)</th>
<th>(\eta = 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MMSE</td>
<td>0.97</td>
<td>0.81</td>
</tr>
<tr>
<td>Color Demon</td>
<td>0.93</td>
<td>0.53</td>
</tr>
</tbody>
</table>

**Critically Damped:**

<table>
<thead>
<tr>
<th>(\eta = 0.1)</th>
<th>(\eta = 1)</th>
<th>(\eta = 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MMSE</td>
<td>0.96</td>
<td>0.78</td>
</tr>
<tr>
<td>Color Demon</td>
<td>0.93</td>
<td>0.48</td>
</tr>
</tbody>
</table>

**Underdamped:**

<table>
<thead>
<tr>
<th>(\eta = 0.1)</th>
<th>(\eta = 1)</th>
<th>(\eta = 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MMSE</td>
<td>0.98</td>
<td>0.89</td>
</tr>
<tr>
<td>Color Demon</td>
<td>0.96</td>
<td>0.77</td>
</tr>
</tbody>
</table>
Uncertainty Principle (quantum mechanics)

- $\sigma_x^2$: position uncertainty variance
- $\sigma_p^2$: momentum uncertainty variance
- $\hbar$: Planck’s constant

\[ \sigma_x \sigma_p \geq \frac{\hbar}{2} \]
Bit Budgeting

- $x$ and $\dot{x}$: state variables
- $B_x$: nats to specify $x$
- $B_{\dot{x}}$: nats to specify $\dot{x}$
- $B_{\text{color}}$: nat budget

$$B_x + B_{\dot{x}} \leq B_{\text{color}}$$
Rate Distortion ↔ Uncertainty
Rate Distortion $\leftrightarrow$ Uncertainty

- Univariate distortion $\mathcal{D}$ is exactly $\sigma_x^2$ or $\sigma_x^2$ ($\sigma_p^2 = M^2 \sigma_x^2$)
Rate Distortion ↔ Uncertainty

- Univariate distortion $\mathcal{D}$ is exactly $\sigma_x^2$ or $\sigma_{\dot{x}}^2$ ($\sigma_p^2 = M^2 \sigma_{\dot{x}}^2$)

- $\sigma^2$: a priori uncertainty of $x$ and $\dot{x}$ so,
Rate Distortion ↔ Uncertainty

• Univariate distortion $D$ is exactly $\sigma_x^2$ or $\sigma_{\dot{x}}^2$ ($\sigma_p^2 = M^2 \sigma_{\dot{x}}^2$)

• $\sigma^2$: a priori uncertainty of $x$ and $\dot{x}$ so,

\[
-\frac{1}{2} \log D + \log \sigma = B \quad \Rightarrow \quad \sqrt{D} = \sigma e^{-B}
\]

\[
\sigma_x \sigma_p = \frac{\sigma^2}{M} e^{-(Bx + B\dot{x})} \geq \frac{\sigma^2}{M} e^{-B_{\text{color}}} = \frac{\sigma^2}{M} e^{-\frac{\eta}{2\lambda_{\text{min}}}}
\]
Rate Distortion ↔ Uncertainty

- Univariate distortion $\mathcal{D}$ is exactly $\sigma_x^2$ or $\sigma_{\dot{x}}^2$ ($\sigma_p^2 = M^2 \sigma_{\dot{x}}^2$)

- $\sigma^2$: a priori uncertainty of $x$ and $\dot{x}$ so,

$$-\frac{1}{2} \log \mathcal{D} + \log \sigma = B \implies \sqrt{\mathcal{D}} = \sigma e^{-B}$$

$$\sigma_x \sigma_p = \frac{\sigma^2}{M} e^{-(Bx + B\dot{x})} \geq \frac{\sigma^2}{M} e^{-B_{\text{color}}} = \frac{\sigma^2}{M} e^{-\frac{\eta}{2\lambda_{\text{min}}}}$$

- NOT claiming equivalence with Uncertainty Principle (yet 😊)
Punchline
Punchline

All
Punchline

All
Physical Interaction
Punchline

All

Physical Interaction

Is
Punchline

All Physical Interaction Is Communication
Punchline

All Physical Interaction Is Communication

SO,
Communication/Information Theorists RULE!