Fundamental Limits of Molecular Communication

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A Simple Statement of Fact
A Simple Statement of Fact

EVERYTHING
A Simple Statement of Fact

EVERYTHING is
A Simple Statement of Fact

EVERYTHING

is

Communication Theory
Followed That Hammer Into Outer Space
“Inscribed Matter” Led To Inner Space
“Inscribed Matter” Led To Inner Space

- **20 lb paper @ 1000dpi**: $2 \times 10^{10}$ bits/kg
- **DVD**: $3 \times 10^{12}$ bits/kg
- **Magnetic Storage** with FeO$_2$: $2 \times 10^{17}$ bits/kg
- **Optical Lithography** with SiO$_2$: $3.85 \times 10^{18}$ bits/kg
- **E-beam Lithography** with SiO$_2$: $1.54 \times 10^{21}$ bits/kg
- **STM** with Xe on Ni: $1.74 \times 10^{22}$ bits/kg
- **RNA**: $3.6 \times 10^{24}$ bits/kg
- **Li + Be**: $7.5 \times 10^{25}$ bits/kg
What Is A ...
What Is A ... Signaling Molecule
A REALLY Simple Signaling Molecule (Token)

Naked (and clothed) Ca^{++}
A Simple Signaling Molecule (Token)

Acyl Homoserine Lactone

Quorum sensing signal
A More Complex Signaling Molecule (Token)

Nerve Growth Factor (protein)
What Is A ...
What Is A ...

Signal Receptor
Receptor Specificity Cartoon

Ligand (token) docks with receptor (protein)
A More Detailed Receptor Specificity Cartoon

Ligands (tokens) dock with receptor (protein)
Wireless With Molecules

Preamble

What Are Some ...
Preamble

What Are Some ...

Communication Examples
Wireless With Molecules

**Reception and Transduction Cartoon**

1. Receptor-ligand binding
2. Signal transduction (via second messengers)
3. Cellular responses
4. Changes in gene expression

**Ligand → Receptor → Gene Tickling**
Identical Tokens: bacteria
Identical Tokens: neurons

ACh release → postsynaptic uptake
Tokens with Payloads: transcription

Nuclear DNA $\rightarrow$ mRNA $\rightarrow$ Ribosome $\rightarrow$ Protein
Active Transport

Bacterial Microtubules
Setup + Punchline Preview
Setup + Punchline Preview

TIMING is FUNDAMENTAL
Setup + Punchline Preview

**TIMING is FUNDAMENTAL**

A game of release (time $t$) and catch (time $s = t + d$)
Setup + Punchline Preview

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Multiple identical molecules: $t \rightarrow s \rightarrow \vec{s}$
Setup + Punchline Preview

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Molecules with embedded payloads (similar math)
Setup + Punchline Preview

**TIMING is FUNDAMENTAL**

A game of release (time $t$) and catch (time $s = t + d$)

Multiple identical molecules: $t \rightarrow s \rightarrow \vec{s}$

Molecules with embedded payloads (similar math)

**OUTRAGEOUSLY Low Power**
Diffusion Cartoon
Diffusion Cartoon

\[ t = t_1 \]
Diffusion Cartoon

\[ t = t_1 \]
Diffusion Cartoon

$t = t_1$
Diffusion Cartoon

\[ t = t_1 \]
Diffusion Cartoon

\[ t = t_1 \]
Diffusion Cartoon

\[ t = t_1 \]
Diffusion Cartoon

\[ t = t_1 \]
Diffusion Cartoon

$t = t_1$

Diagram of diffusion process at time $t_1$. The solid line represents the initial boundary, and the open line indicates the boundary at a later time.
Diffusion Cartoon

$t = t_1$
Diffusion Cartoon

\[ t = t_1 \]
Diffusion Cartoon

$t = t_1$
Diffusion Cartoon

\[ t = t_1 \]

\[ s = t_1 + d \]
Transport (passive) Receptor Kinetics (ignore)
Transport (passive) Receptor Kinetics (ignore)

Coding → Emission → Transport → Capture → Decoding
Could Even Add Some Drift

Coding $\rightarrow$ Emission $\rightarrow$ Transport $\rightarrow$ Capture $\rightarrow$ Decoding
Mathematical Abstraction For Identical Tokens
Mathematical Abstraction For Identical Tokens

\[ S = T + D \]
Mathematical Abstraction For Identical Tokens

\[ S = T + D \]

\[ S = \text{Sort}[S] \]
Mathematical Abstraction For Identical Tokens

\[ S = T + D \]
\[ S = \text{Sort}[S] \]

First passage time: \( E[D] = 1/\mu \)
Token Timing

Departures

Arrivals

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Mutual Information: $I(S; T)$

$M$ tokens on an interval $\tau(M)$
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$M$ tokens on an interval $\tau(M)$

$$I(S; T) = h(S) - h(S|T)$$

$$= h(S) - h(D)$$

$$\leq M (h(S) - h(D)), \quad (\text{i.i.d. } D)$$
**Mutual Information:** \( I(S; T) \)

\( M \) tokens on an interval \( \tau(M) \)

\[
I(S; T) = h(S) - h(S|T) \\
= h(S) - h(D) \\
\leq M (h(S') - h(D)) , \quad \text{(i.i.d. } D) \\
\]

\textbf{Max } h(S') , Done!
**Mutual Information:** $I(S; T)$

$M$ tokens on an interval $\tau(M)$

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I(S; T) = h(S) - h(S|T)
= h(S) - h(D)
\leq M (h(S) - h(D)) ,
\]

(i.i.d. $D$)

**Max $h(S)$, Done!**

**Easy, Right!?!**
Mutual Information: $I(S; T)$

$M$ tokens on an interval $\tau(M)$

$I(S; T) = h(S) - h(S|T)$

$= h(S) - h(D)$

$\leq M (h(S) - h(D))$, (i.i.d. $D$)

Max $h(S)$, Done!

Easy, Right!?!}$I(\vec{S}; T) = h(\vec{S}) - h(\vec{S}|T) =$?
Hypersymmetries

Departures

Arrivals
Hypersymmetries

Departures

Arrivals
Hypersymmetry Buys You
Hypersymmetry Buys You

\[ h(\tilde{S}) = h(S) - \log M! \]
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\( \{ \tilde{S}, \Omega \} \leftrightarrow S \)
Hypersymmetry Buys You

\[ h(\tilde{S}) = h(S) - \log M! \]

\[ \{\tilde{S}, \Omega\} \leftrightarrow S \]

\[ h(\tilde{S}|T) = H(\Omega|\tilde{S}, T) - h(S|T) \]
**Hypersymmetry Buys You**

\[ h(\vec{S}) = h(S) - \log M! \]

\[
\{ \vec{S}, \Omega \} \leftrightarrow S \\
\downarrow \\
\]

\[ h(\vec{S}|T) = H(\Omega|\vec{S}, T) - h(S|T) \]

\[
I(\vec{S}; T) = h(S) + H(\Omega|\vec{S}, T) - (\log M! + h(D)) \\
\text{The Money!} - \text{constant} 
\]
Channel Use Formalities
Channel Use Formalities

Guard Interval: \( \gamma(M, \epsilon) \)  
Overflow Probability: \( \epsilon \)
Channel Use Formalities

Guard Interval: $\gamma(M, \epsilon)$  Overflow Probability: $\epsilon$

Power Constraint (tokens cost energy):

$$\rho \equiv \lim_{\epsilon \to 0} \lim_{M \to \infty} \frac{M}{\tau(M) + \gamma(M, \epsilon)}$$
Limiting Details
Limiting Details

Set: $\gamma(M, \epsilon) = \epsilon \tau(M)$ (convenience)
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Require: \( \lim_{M \to \infty} \text{Prob}\{\tilde{S}_M \leq \tau(M)(1 + \epsilon)\} = 1 \)
Limiting Details

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Require: \( \lim_{M \to \infty} \text{Prob}\{\bar{S}_M \leq \tau(M)(1 + \epsilon)\} = 1 \)

Worst case: all tokens launched at time \( \tau(M) \)
**Limiting Details**

**Set:** \( \gamma(M, \epsilon) = \epsilon \tau(M) \) (convenience)

**Require:** \( \lim_{M \to \infty} \text{Prob}\{\vec{S}_M \leq \tau(M)(1 + \epsilon)\} = 1 \)

**Worst case:** all tokens launched at time \( \tau(M) \)

**PUNCHLINE:** all ok if \( E[D] \) exists
Omitting the Details (or summary :) )
Omitting the Details (or summary :) )

Set: \( \rho \equiv \frac{M}{\tau(M)} \)

Require: \( E[D] < \infty \)

Define: \( \chi \equiv \frac{\mu}{\rho} \) (first passage rate) \( \rho \) (token launch rate)

\[
C_m(M) = \max_{\text{hypersymm } f_T(\cdot)} \left( \frac{I(\bar{S}; T)}{M} \right)
\]

\[
C_m = \lim_{M \to \infty} C_m(M)
\]

\[
C_t = \rho C_m
\]
My Past Personal Struggles
My Past Personal Struggles

∃ closed form results/bounds for $H(\Omega|\vec{S}, T)$
My Past Personal Struggles

∃ closed form results/bounds for $H(\Omega|\vec{S}, T)$

$\max_{f_T()} h(S) + H(\Omega|\vec{S}, T) \geq ?$ (ISIT’13)
My Past Personal Struggles

∃ closed form results/bounds for $H(\Omega | \vec{S}, T)$

$\max_{f_T()} h(S) + H(\Omega | \vec{S}, T) \geq ?$ (ISIT’13)

$\max_{f_T()} h(S) + H(\Omega | \vec{S}, T) \leq ?$ (ISIT’14)
Timing + Payload
Timing + Payload

Identical tokens $\rightarrow$ timing info only
Timing + Payload

Identical tokens $\rightarrow$ timing info only

Payloads $\rightarrow$ chop message into $M \cdot B$-bit pieces
Timing + Payload

Identical tokens $\rightarrow$ timing info only

Payloads $\rightarrow$ chop message into $M \cdot B$-bit pieces

**BUT**: Payloads can arrive out of order
Timing + Payload

Identical tokens $\rightarrow$ timing info only

Payloads $\rightarrow$ chop message into $M$ $B$-bit pieces

BUT: Payloads can arrive out of order

Add $H(\Omega|\vec{S}, T)/M$ bits per token

(for re-sequencing)
Energy
Energy

Identical Tokens: $c_0$ joules per token
Energy

Identical Tokens: $c_0$ joules per token

Inscribed Tokens:
Energy

Identical Tokens: $c_0$ joules per token

Inscribed Tokens:

- substrate: $c_1$ joules per token
- payload bit $B$: $B \Delta c_1$ joules per token
- avg. sequence bits $K$: $K \Delta c_1$ joules per token, so
**Energy**

**Identical Tokens:** $c_0$ joules per token

**Inscribed Tokens:**

- **substrate:** $c_1$ joules per token
- **payload bit $B$:** $B\Delta c_1$ joules per token
- **avg. sequence bits $K$:** $K\Delta c_1$ joules per token, so

$$H(\Omega|\vec{S}, T) \leq MK \leq \log M!$$
And Now ...
And Now ...

LOWER BOUNDS

using exponential first passage
(the timing channel’s “Gaussian”)

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Theorem 1.

\[ C_T \geq \frac{1}{c_0} \left( \log \chi + e^{-\frac{1}{\chi}} \sum_{k=2}^{\infty} \left( \frac{1}{\chi} \right)^k (k\chi - 1) \frac{\log k!}{k!} \right) \]

\[ H(\Omega|\bar{S}, T)/M : \text{average per-token order-uncertainty} \]
Payload-Only Bits/Joule

**Theorem 2.**

\[ C_P = \frac{B}{c_1 + \Delta c_1 \left( B + \min_t \frac{1}{M} H(\Omega|\vec{S}, t) \right)} \]

**Theorem 3.**

\[ C_P \geq \frac{B}{c_1 + \Delta c_1 \left( B + e^{-\frac{1}{\chi}} \sum_{k=2}^{\infty} \left( \frac{1}{\chi} \right)^k (k\chi - 1) \frac{\log k!}{k!} \right) \left( H(\Omega|\vec{S}, T)/M : \text{average per-token order-uncertainty} \right)} \]
Payload + Timing Bits/Joule Lower Bound

Theorem 4.

\[ \mathcal{R}_{P+T} \approx c_1 + \Delta c_1 \left( \log \left( 1 + \frac{\chi M}{e} \right) + B \right) \]

\[ B + e^{-\frac{1}{\chi}} \sum_{k=2}^{\infty} \left( \frac{1}{\chi} \right)^k (k\chi - 1) \frac{\log k!}{k!} \]

where \( \mathcal{R}_{P+T} \leq \mathcal{C}_{P+T} \).
Info per Unit Energy

χ ↔ passage rate per launch rate

\[ c_0 = 1, \ c_1 = 0, \ \Delta c_1 = 1 \]
\[ \frac{1}{\chi} \leftrightarrow \text{launch rate per passage rate} \]
\[ c_0 = 1, \ c_1 = 0, \ \Delta c_1 = 1 \]
And Now ....
And Now ....

Numerical Play Time
Play Time Setup
Play Time Setup

source \quad R \quad sink
Play Time Setup

source \quad R \quad sink

“Binary Protein” Token Construction $4B\text{ATP} = 3.2B \times 10^{-19} J$
Play Time Setup

“Binary Protein” Token Construction \( 4B \text{ATP} = 3.2B \times 10^{-19} J \)

Diffusion Coefficient, \( \mathcal{D} \) in air: \( \approx 10^{-5} m^2/s \)

Mean First Passage Time, \( E[D] \approx \frac{R^2}{2D} \)
Play Time Setup

source \[ \text{R} \] sink

“Binary Protein” Token Construction \( 4B \text{ATP} = 3.2B \times 10^{-19} \text{J} \)

Diffusion Coefficient, \( D \) in air: \( \approx 10^{-5} \text{m}^2/\text{s} \)

Mean First Passage Time, \( E[D] \approx \frac{R^2}{2D} \)

Across a table (1m): \( E[D] \approx 14 \text{hrs} \) (need fan 😊)
“Binary Protein” Token Construction \(4B\text{ATP} = 3.2B \times 10^{-19} J\)

Diffusion Coefficient, \(D\) in air: \(\approx 10^{-5} m^2/s\)
Mean First Passage Time, \(E[D] \approx \frac{R^2}{2D}\)

Across a table (1m): \(E[D] \approx 14 hrs\) (need fan 😊)

Across a 0.1mm gap: \(E[D] = 0.5 ms\)
Play Time Numbers
Play Time Numbers

\[ \frac{1}{\chi} = \frac{\rho}{\mu} = 1 \text{ (w/ identical tokens)} \]
Play Time Numbers

\[ \frac{1}{\chi} = \frac{\rho}{\mu} = 1 \] (w/ identical tokens)

Across a table: \( \approx 2 \) bits/day (\( \approx 7 \times 10^{-24} \) W)

Across a 0.1mm gap: \( \approx 10 \) kb/s (\( \approx 3.2 \) fW)
Play Time Numbers

\[ \frac{1}{\chi} = \frac{\rho}{\mu} = 1 \quad \text{(w/ identical tokens)} \]

Across a table: \( \approx 2 \text{ bits/day} \quad (\approx 7 \times 10^{-24} \text{ W}) \)

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\[ \frac{1}{\chi} = \frac{\rho}{\mu} = 1000 \quad \text{(w/ } B = 1000\text{-bit tokens)} \]
Play Time Numbers

\[ \frac{1}{\chi} = \frac{\rho}{\mu} = 1 \quad (w/ \text{identical tokens}) \]

**Across a table:** \( \approx 2 \text{ bits/day} \) (\( \approx 7 \times 10^{-24} \text{ W} \))

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\[ \frac{1}{\chi} = \frac{\rho}{\mu} = 1000 \quad (w/ \ B = 1000\text{-bit tokens}) \]

**Across a table:** \( \approx 2\text{Kb/day} \) (\( \approx 7 \times 10^{-21} \text{ W} \))

**Across a 0.1mm gap:** \( \approx 10\text{Mb/s} \) (\( \approx 3.2 \text{ pW} \))
Play Time Numbers

\[ \frac{1}{\chi} = \frac{\rho}{\mu} = 1 \text{ (w/ identical tokens)} \]

Across a table: \( \approx 2 \text{ bits/day (}\approx 7 \times 10^{-24} \text{ W)} \)
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\[ \frac{1}{\chi} = \frac{\rho}{\mu} = 1000 \text{ (w/ } B = 1000\text{-bit tokens)} \]

Across a table: \( \approx 2\text{Kb/day (}\approx 7 \times 10^{-21} \text{ W)} \)
Across a 0.1mm gap: \( \approx 10\text{Mb/s (}\approx 3.2 \text{ pW)} \)

fiber: \( (100\text{Tb/s@0.2W}) \ 5 \times 10^{14}\text{bits/J} \)

molecule: \( \approx 3 \times 10^{18}\text{bits/J} \)
Appropriately Awed Response
Netflix/SensorNet Distribution Fantasy
Disk Farm Fantasy

Suppose token construction energy cost \ll\text{fan energy cost}
Disk Farm Fantasy

Suppose token construction energy cost ≪ fan energy cost

\[ 1 \mu g \text{ RNA per second} \Rightarrow 3.6 \times 10^{15} \text{ bits/sec} \]
Molecular Communication Summary
Molecular Communication Summary

Timing is THE MOST Fundamental Treatment
Molecular Communication Summary

Timing is THE MOST Fundamental Treatment

Need Bit Efficiency?
Molecular Communication Summary

Timing is THE MOST Fundamental Treatment

Need Bit Efficiency?
Slow release with timing &/or small payload
Molecular Communication Summary

Timing is THE MOST Fundamental Treatment

Need Bit Efficiency?
Slow release with timing &/or small payload

Need Rate Efficiency?
Molecular Communication Summary

Timing is THE MOST Fundamental Treatment

Need Bit Efficiency?
Slow release with timing &/or small payload

Need Rate Efficiency?
Fast release with payload + timing or large payload
Molecular Communication Summary

Timing is THE MOST Fundamental Treatment

Need Bit Efficiency?
Slow release with timing &/or small payload

Need Rate Efficiency?
Fast release with payload + timing or large payload

Scary Efficiencies
Molecular Communication Summary

Timing is THE MOST Fundamental Treatment

Need Bit Efficiency?
Slow release with timing &/or small payload

Need Rate Efficiency?
Fast release with payload + timing or large payload

Scary Efficiencies
(beware transport latency)
If You Only Remember One Slide

A truck filled with storage media, driven across town, is a very reliable high bit rate channel.

–Comm. Theory Collective Subconscious

BUT ...

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If You Only Remember One Slide

*A truck filled with storage media, driven across town, is a very reliable high bit rate channel.*

—Comm. Theory Collective Subconscious

**BUT ...**

*A swarm of timed gnats*
If You Only Remember One Slide

A truck filled with storage media, driven across town, is a very reliable high bit rate channel.

–Comm. Theory Collective Subconscious

BUT ...

A swarm of timed gnats with backpacks
If You Only Remember One Slide

A truck filled with storage media, driven across town, is a very reliable high bit rate channel.

–Comm. Theory Collective Subconscious

BUT ...

A swarm of timed gnats with backpacks in a breeze
If You Only Remember One Slide

*A truck filled with storage media, driven across town, is a very reliable high bit rate channel.*

—Comm. Theory Collective Subconscious

**BUT ...**

*A swarm of timed gnats with backpacks in a breeze could be better.*