Fundamental Limits of Molecular Communication

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A Simple Statement of Fact

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EVERYTHING

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is

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is

Communication Theory

Followed That Hammer Into Outer Space

“Inscribed Matter” Led To Inner Space

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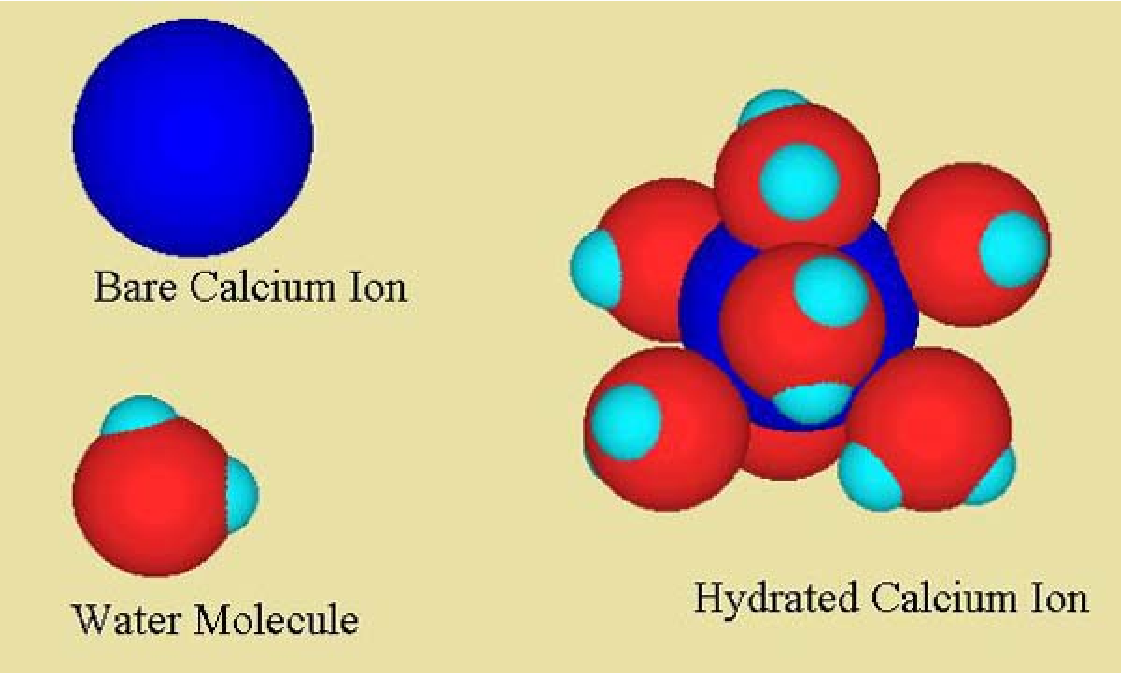
- **20 lb paper @** 1000dpi: 2×10^{10} bits/kg
- **DVD**: 3×10^{12} bits/kg
- **Magnetic Storage** with FeO_2: 2×10^{17} bits/kg
- **Optical Lithography** with SiO_2: 3.85×10^{18} bits/kg
- **E-beam Lithography** with SiO_2: 1.54×10^{21} bits/kg
- **STM** with Xe on Ni: 1.74×10^{22} bits/kg
- **RNA**: 3.6×10^{24} bits/kg
- **Li + Be**: 7.5×10^{25} bits/kg

What Is A ...

What Is A ...

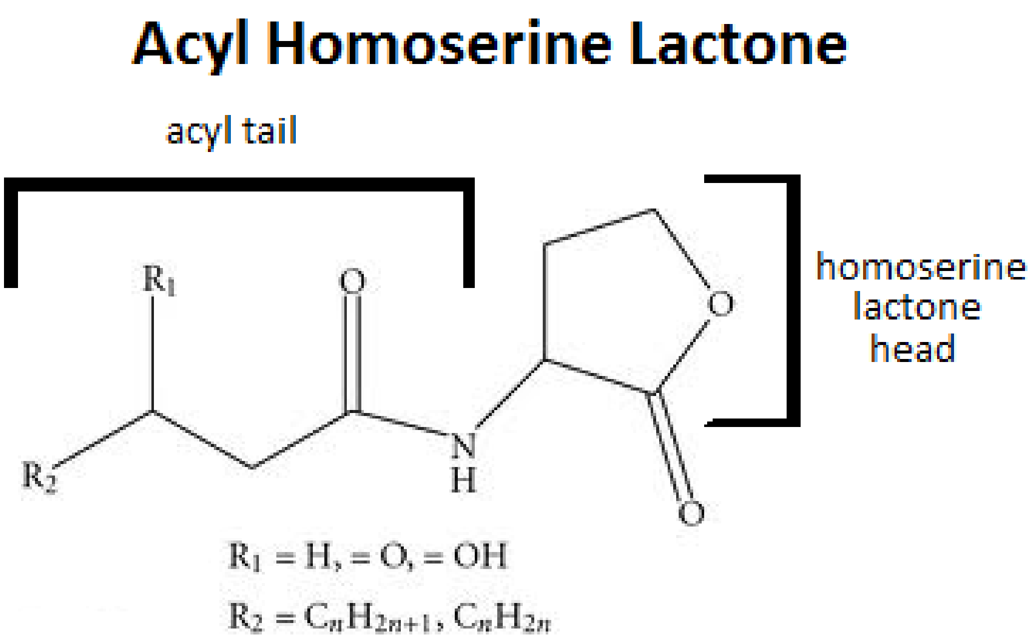
Signaling Molecule

A REALLY Simple Signaling Molecule (Token)



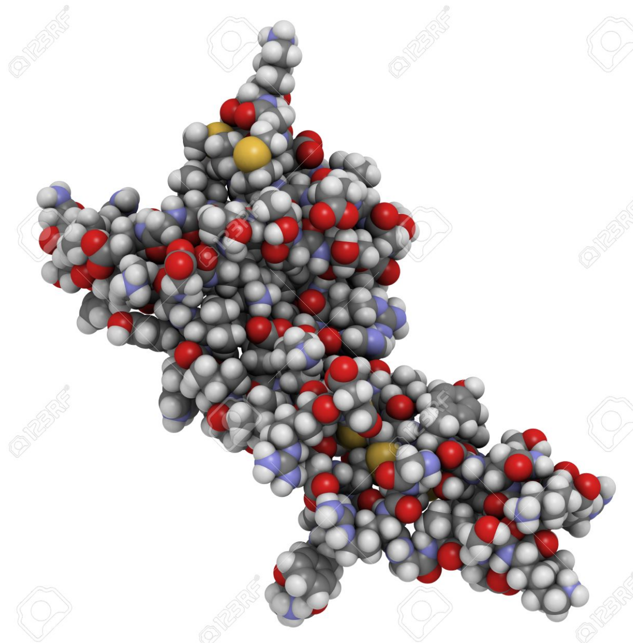
Naked (and clothed) Ca^{++}

A Simple Signaling Molecule (Token)



Quorum sensing signal

A More Complex Signaling Molecule (Token)



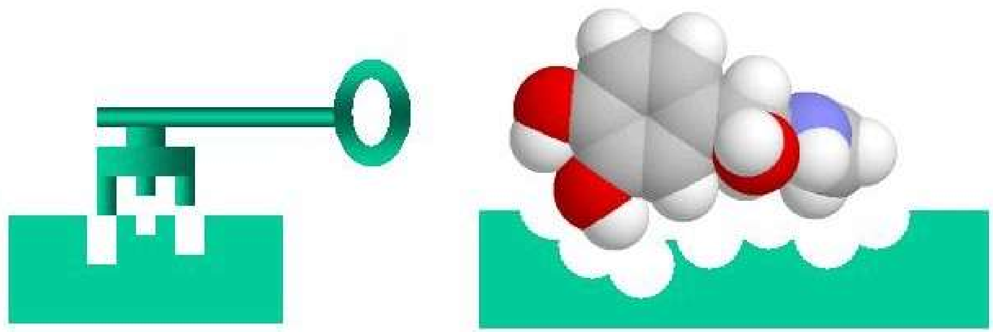
Nerve Growth Factor (protein)

What Is A ...

What Is A ...

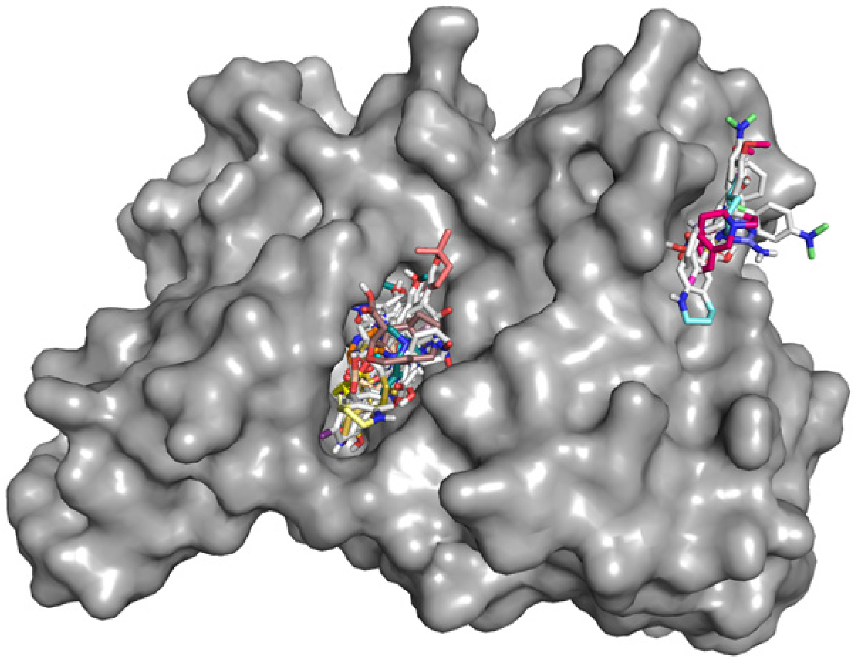
Signal Receptor

Receptor Specificity Cartoon



Ligand (token) docks with receptor (protein)

A More Detailed Receptor Specificity Cartoon



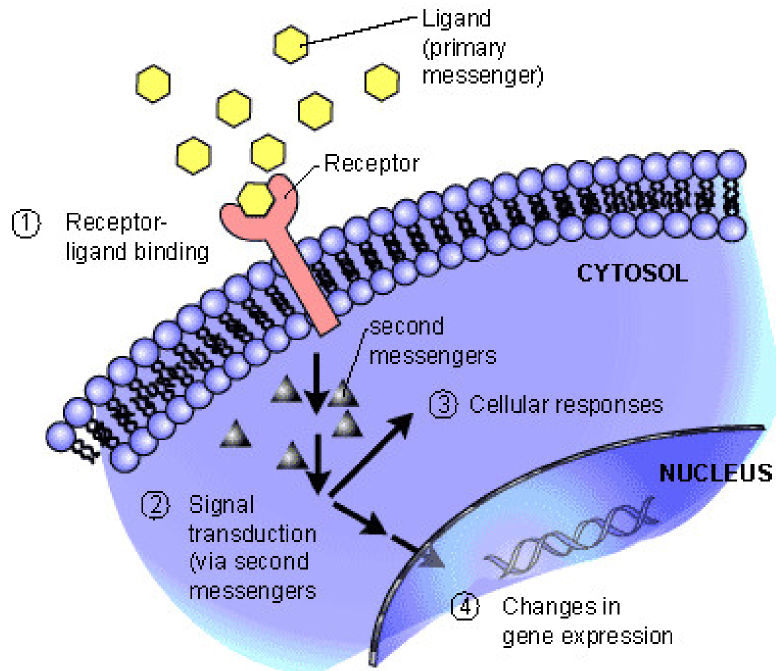
Ligands (tokens) dock with receptor (protein)

What Are Some ...

What Are Some ...

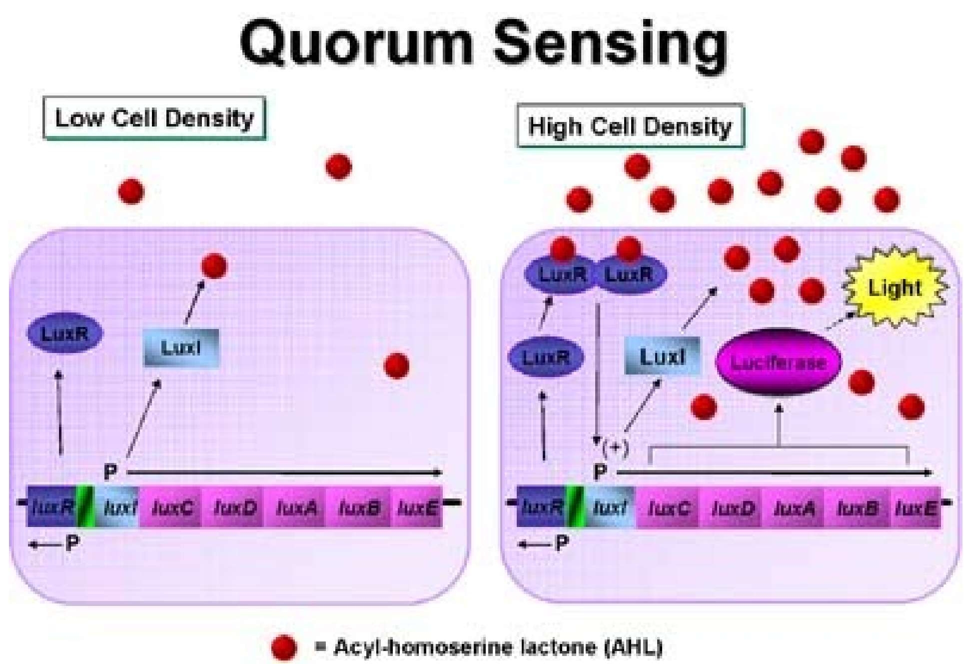
Communication Examples

Reception and Transduction Cartoon

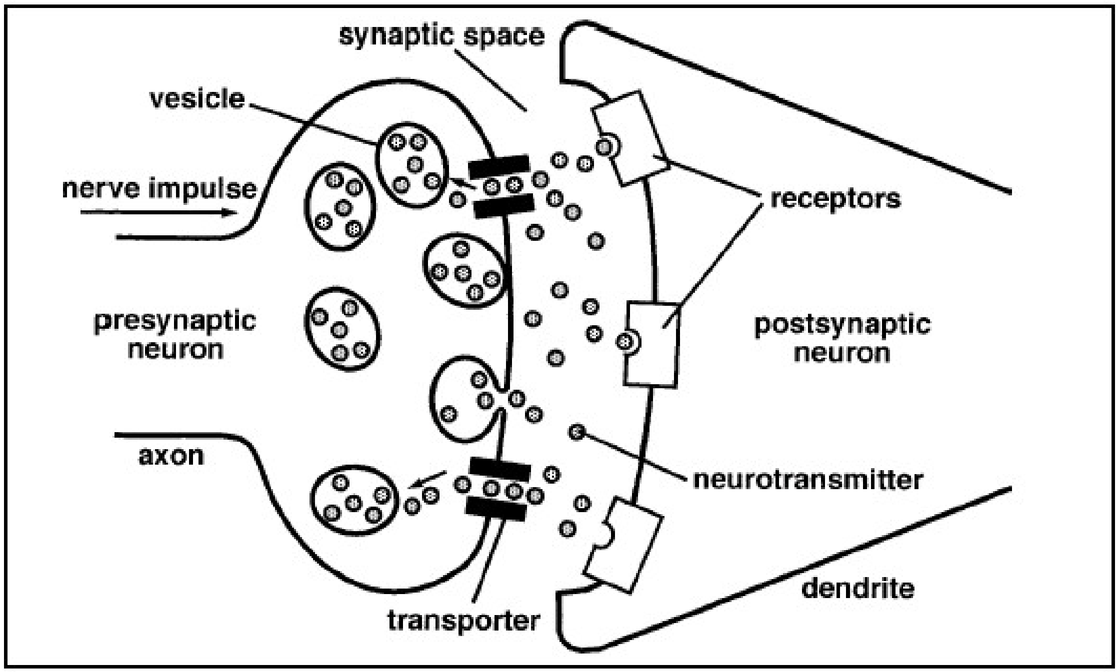


Ligand → Receptor → Gene Tickling

Identical Tokens: bacteria

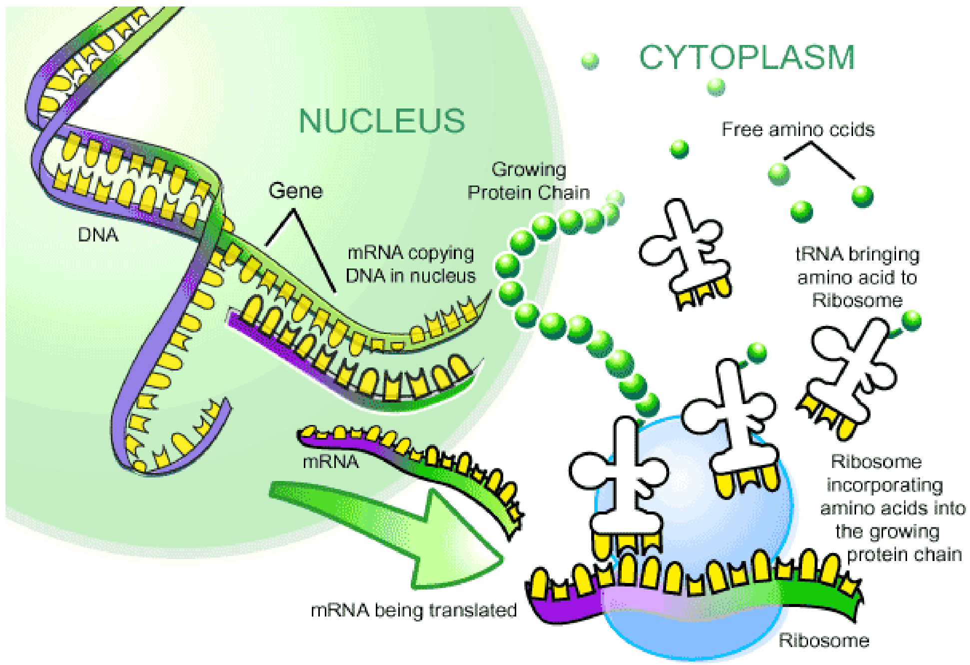


Identical Tokens: neurons



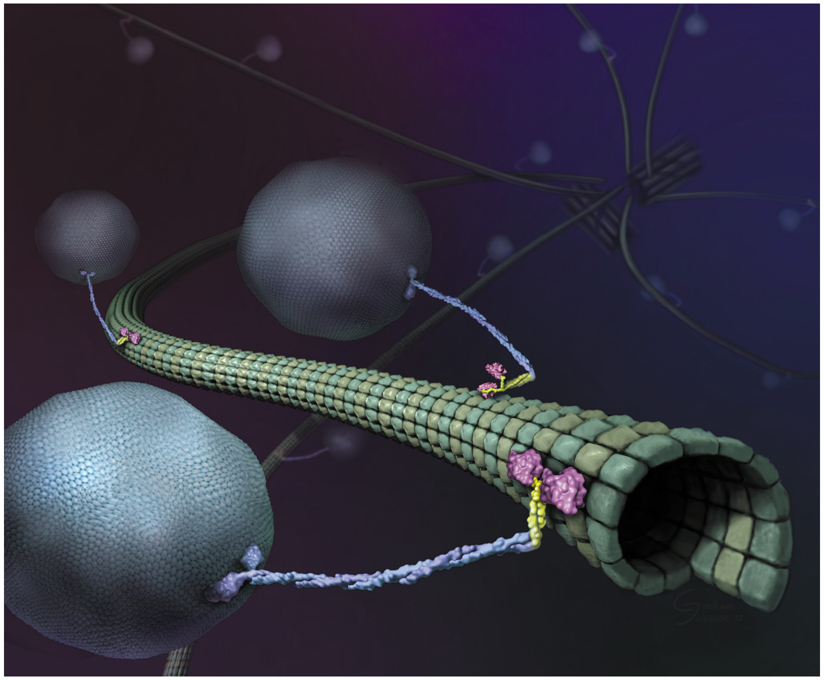
ACh release $\rightarrow$ postsynaptic uptake

Tokens with Payloads: transcription



Nuclear DNA $\rightarrow$ mRNA $\rightarrow$ Ribosome $\rightarrow$ Protein

Active Transport



Bacterial Microtubules

Setup + Punchline Preview

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TIMING is **FUNDAMENTAL**

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A game of release (time t) and catch (time $s = t + d$)

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Molecules with embedded payloads (similar math)

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OUTRAGEOUSLY Low Power

Diffusion Cartoon

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$t = t_1$

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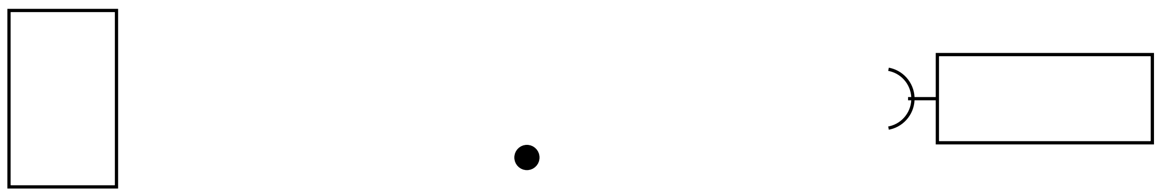

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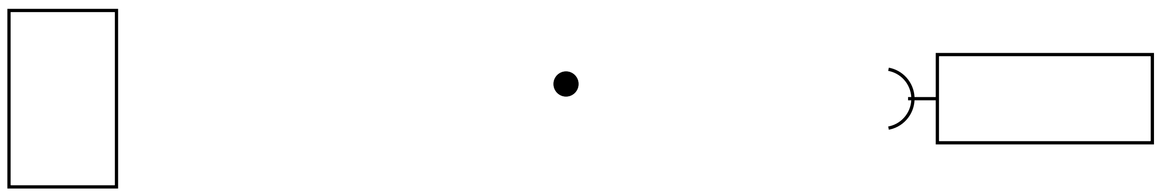

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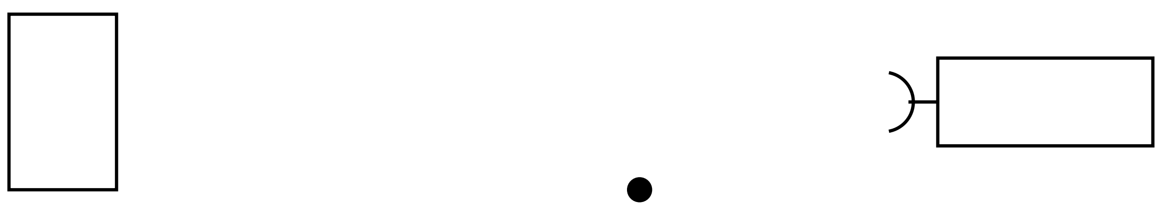

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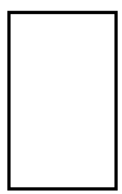
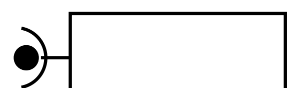


$s = t_1 + d$



Transport (passive) Receptor Kinetics (ignore)

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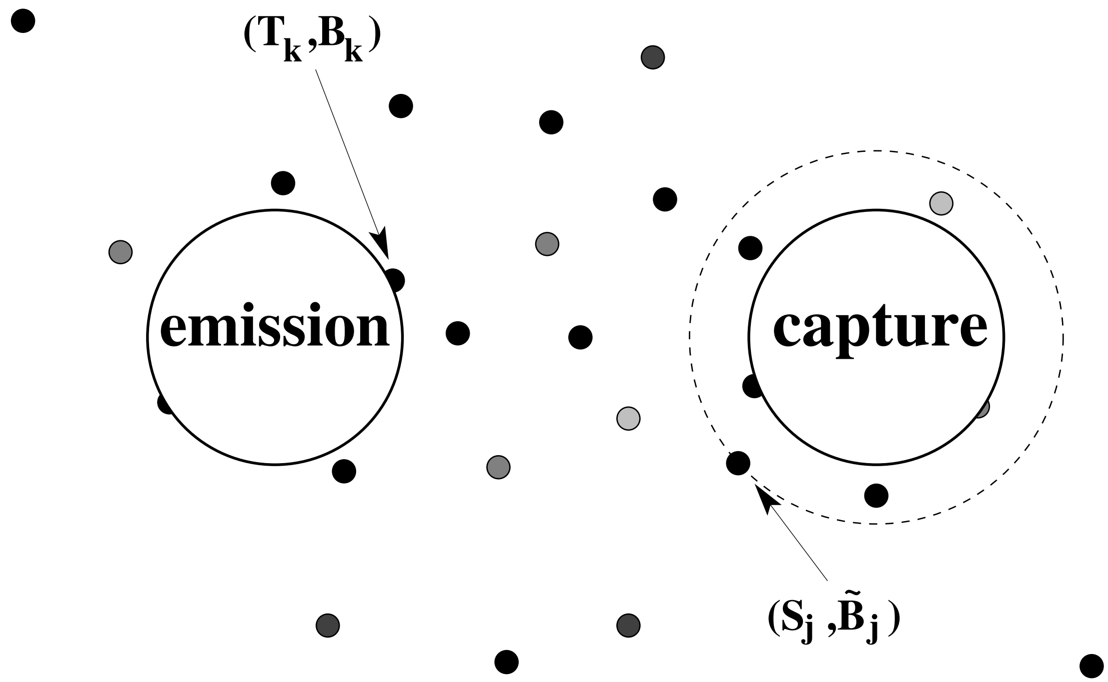
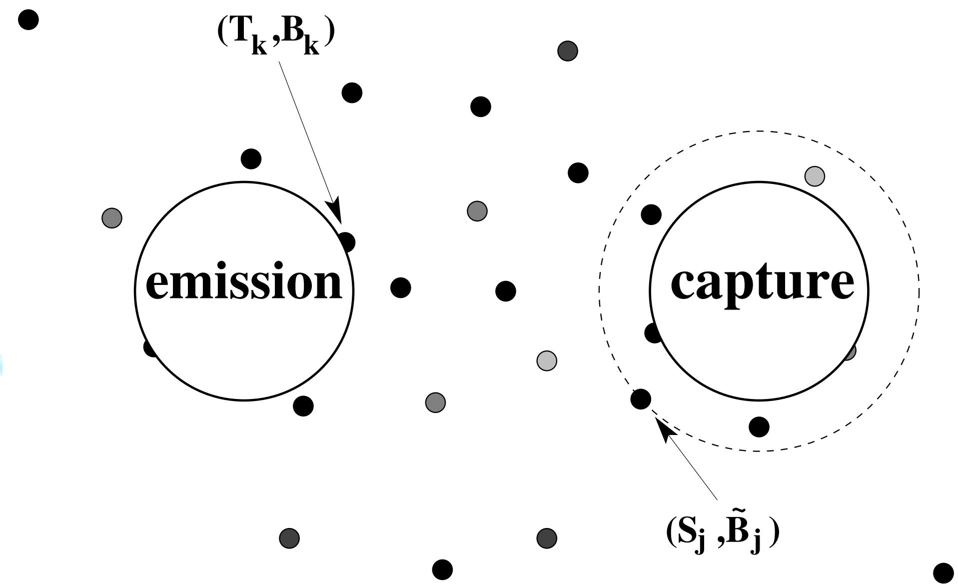


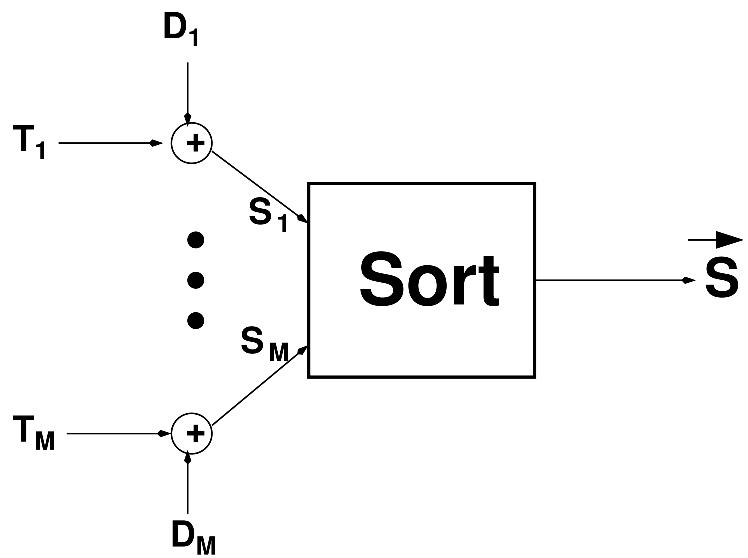
Coding $\rightarrow$ Emission $\rightarrow$ Transport $\rightarrow$ Capture $\rightarrow$ Decoding

Could Even Add Some Drift



Coding → Emission → Transport → Capture → Decoding

Mathematical Abstraction For Identical Tokens



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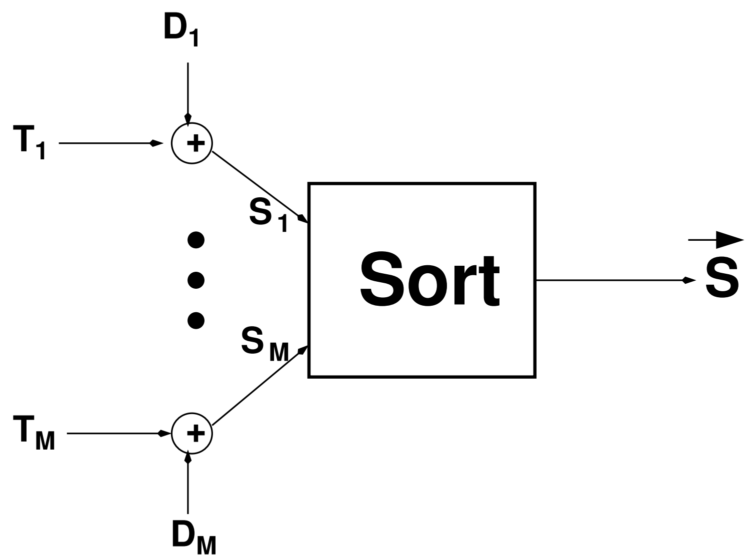

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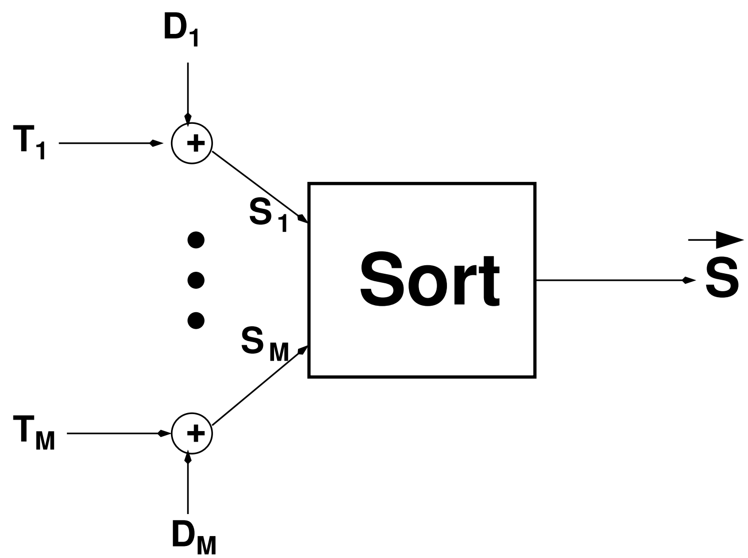

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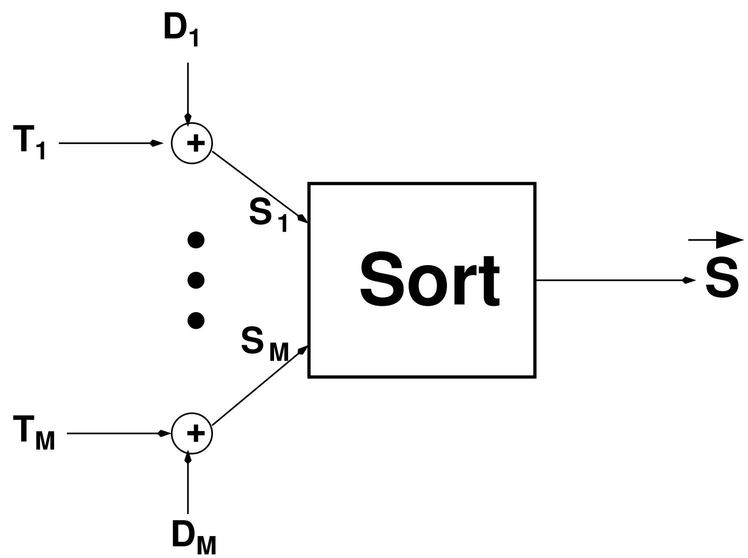

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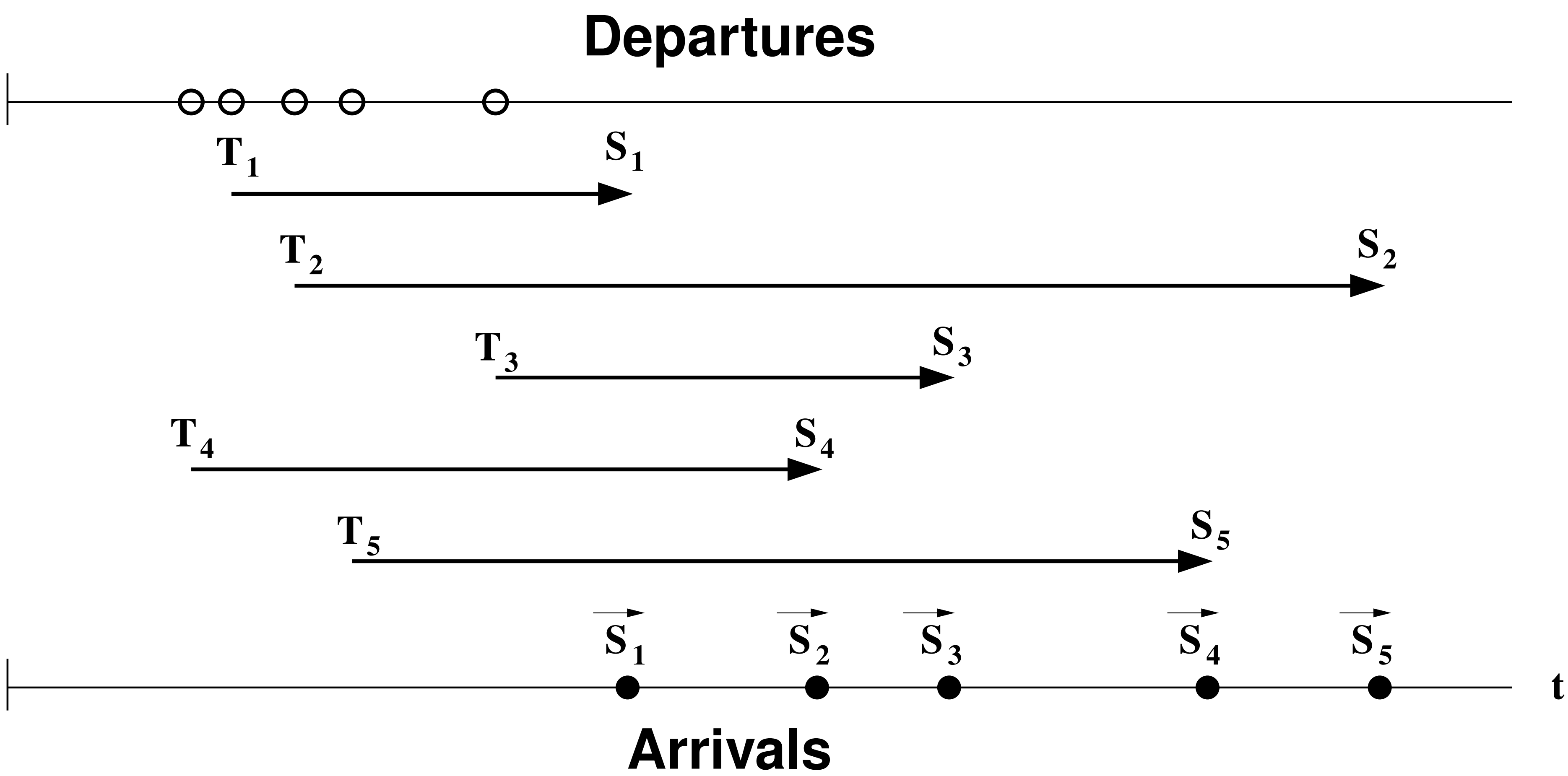
First passage time: $E[D] = 1/\mu$

Token Timing



Mutual Information: $I(\mathbf{S};\mathbf{T})$

M tokens on an interval $\tau(M)$

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Max $h(S)$, Done!

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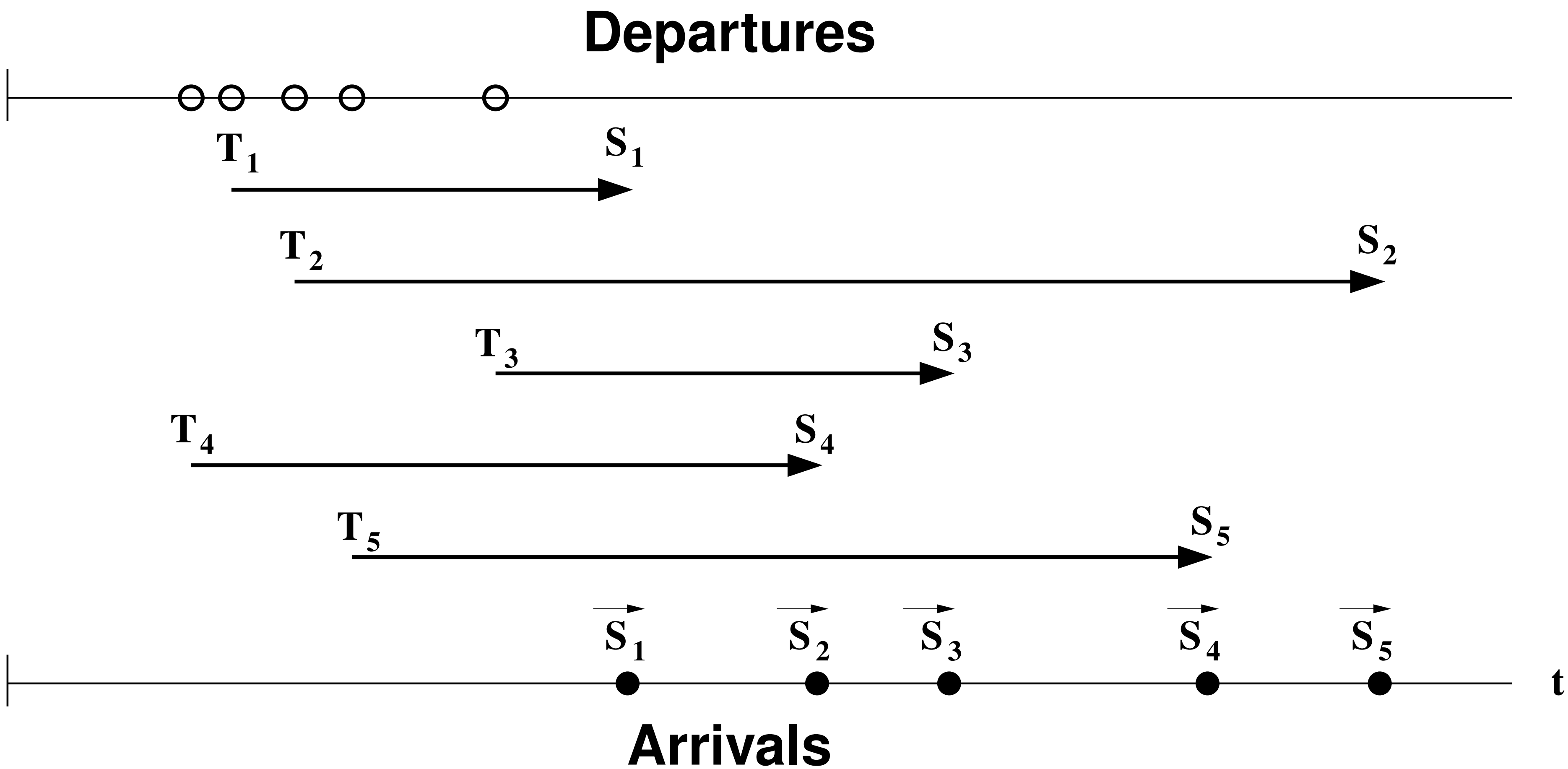
$$I(\vec{\mathbf{S}};\mathbf{T}) = h(\vec{\mathbf{S}}) - h(\vec{\mathbf{S}}|\mathbf{T}) = \,?$$

Hypersymmetries



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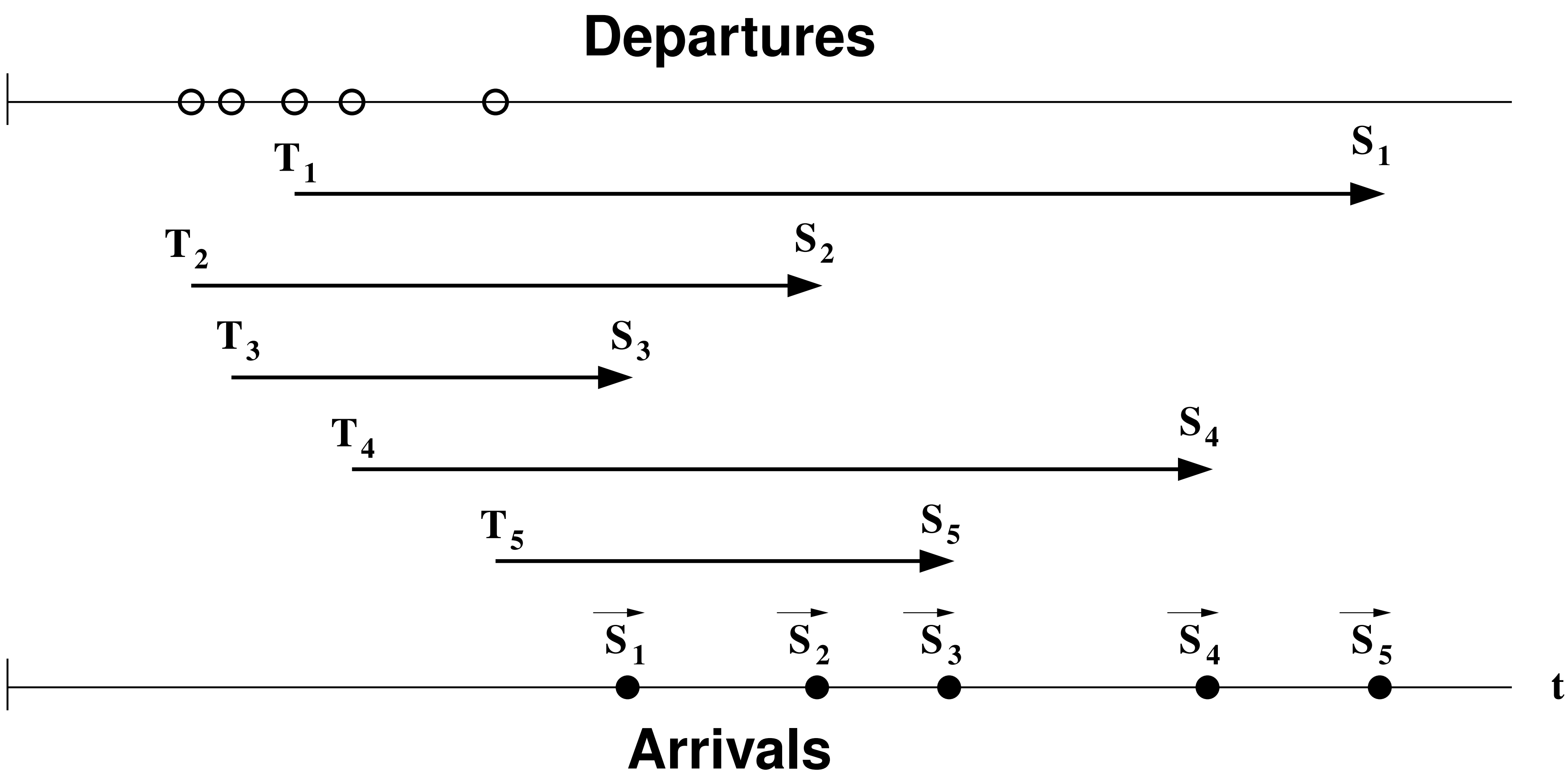

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$$I(\vec{\mathbf{S}}; \mathbf{T}) = \underbrace{h(\mathbf{S}) + H(\Omega|\vec{\mathbf{S}}, \mathbf{T})}_{\text{The Money!}} - \underbrace{(\log M! + h(\mathbf{D}))}_{\text{constant}}$$

Channel Use Formalities

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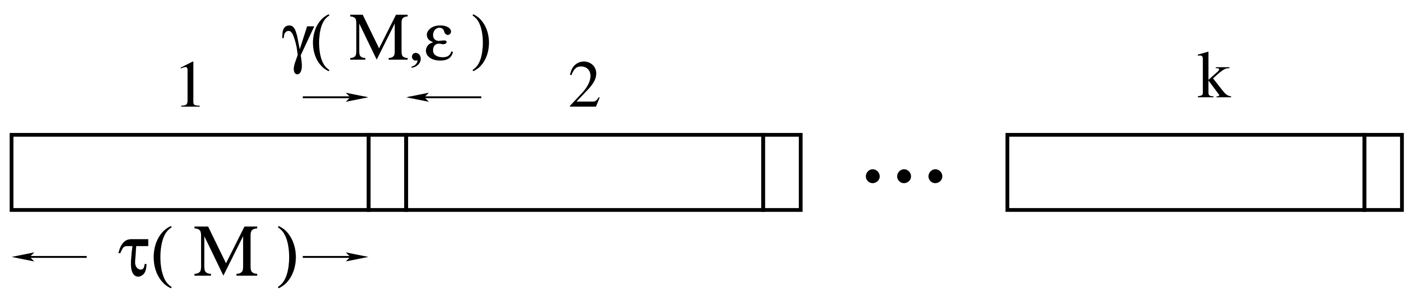

Guard Interval: $\gamma(M, \epsilon)$ Overflow Probability: ϵ

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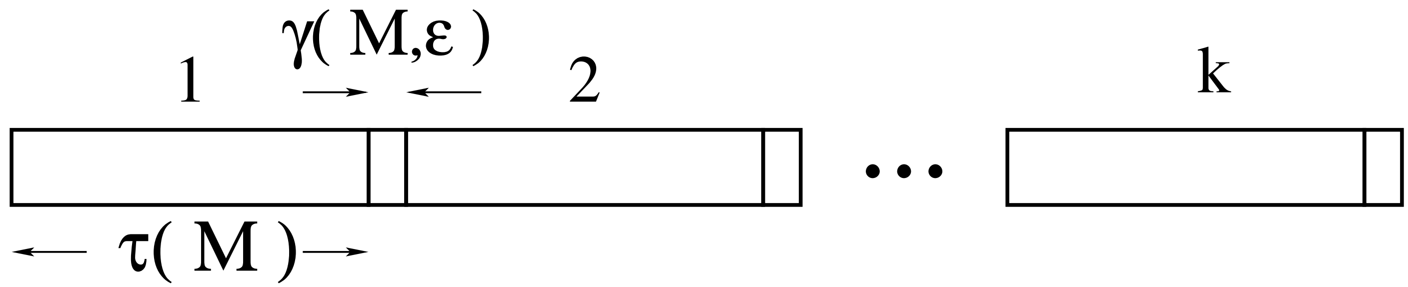

Guard Interval: $\gamma(M,\epsilon)$ Overflow Probability: ϵ

Power Constraint (tokens cost energy):

$$\rho \equiv \lim_{\epsilon \to 0} \lim_{M \to \infty} \frac{M}{\tau(M) + \gamma(M,\epsilon)}$$

Limiting Details

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Set: $\gamma(M, \epsilon) = \epsilon\tau(M)$ (convenience)

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PUNCHLINE: all ok if $E[D]$ exists

Omitting the Details (or summary :))


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Set: $\rho \equiv \frac{M}{\tau(M)}$

Define: $\chi \equiv \frac{\mu \text{ (first passage rate)}}{\rho \text{ (token launch rate)}}$

Require: $E[D] < \infty$

$$C_m(M) = \max_{\text{hypersymm } f_{\mathbf{T}}()} \left(I(\vec{\mathbf{S}}; \mathbf{T})/M \right)$$

$$C_m = \lim_{M \to \infty} C_m(M)$$

$$C_t = \rho C_m$$

My Past Personal Struggles

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- $\exists$ closed form results/bounds for $H(\Omega|\vec{\mathbf{S}}, \mathbf{T})$

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- $$\max_{f_{\mathbf{T}}()} h(\mathbf{S}) + H(\Omega|\vec{\mathbf{S}}, \mathbf{T}) \geq ?$$ (ISIT'13)

My Past Personal Struggles

$\exists$ closed form results/bounds for $H(\Omega|\vec{\mathbf{S}}, \mathbf{T})$

$$\max_{f_{\mathbf{T}}()} h(\mathbf{S}) + H(\Omega|\vec{\mathbf{S}}, \mathbf{T}) \geq \text{?}$$ (ISIT'13)

$$\max_{f_{\mathbf{T}}()} h(\mathbf{S}) + H(\Omega|\vec{\mathbf{S}}, \mathbf{T}) \leq \text{?}$$ (ISIT'14)

Timing + Payload

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BUT: Payloads can arrive out of order

Add $H(\Omega|\vec{\mathrm{S}}, \mathrm{T})/M$ bits per token

(for re-sequencing)

Energy

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Identical Tokens: c_0 joules per token

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Inscribed Tokens:

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Inscribed Tokens:

substrate:	c_1 joules per token
payload bit B**:**	$B\Delta c_1$ joules per token
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Inscribed Tokens:

substrate: c_1 joules per token
payload bit B: $B\Delta c_1$ joules per token
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$$H(\Omega|\vec{\mathbf{S}}, \mathbf{T}) \leq \boldsymbol{MK} \leq \log M!$$

And Now ...

And Now ...

LOWER BOUNDS

using exponential first passage

(the timing channel's “Gaussian”)

Timing-Only Bits/Joule

Theorem 1.

$$\mathcal{C}_T \geq \frac{1}{c_0}\left(\log \chi + \underbrace{e^{-\frac{1}{\chi}}\sum_{k=2}^{\infty}\left(\frac{1}{\chi}\right)^k (k\chi - 1)\frac{\log k!}{k!}}_{H(\Omega|\vec{\mathrm{S}},\mathrm{T})/M\text{: average per-token order-uncertainty}}\right)$$

Payload-Only Bits/Joule

Theorem 2.

$$\mathcal{C}_P = \frac{B}{c_1 + \Delta c_1 \left(B + \min_{\mathbf{t}} \frac{1}{M} H(\Omega | \vec{\mathbf{S}}, \mathbf{t}) \right)}$$

Theorem 3.

$$\mathcal{C}_P \geq \frac{B}{c_1 + \Delta c_1 \left(B + \underbrace{e^{-\frac{1}{\chi}} \sum_{k=2}^{\infty} \left(\frac{1}{\chi} \right)^k (k\chi - 1) \frac{\log k!}{k!}}_{H(\Omega | \vec{\mathrm{S}}, \mathrm{T}) / M \text{: average per-token order-uncertainty}} \right)}$$

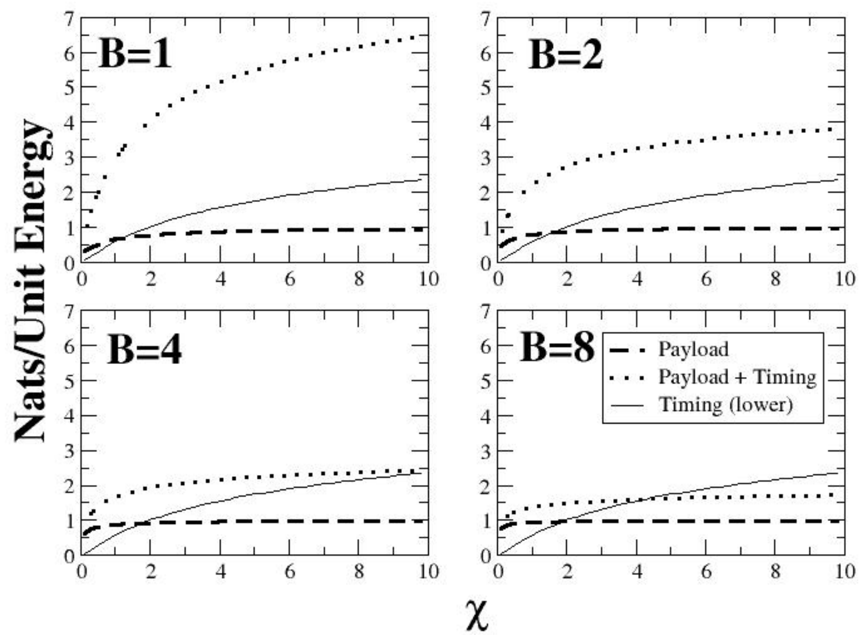
Payload + Timing Bits/Joule Lower Bound

Theorem 4.

$$\mathcal{R}_{P+T} \approx \frac{\log\left(1+\frac{\chi M}{e}\right)+B}{c_1+\Delta c_1\left(B+\underbrace{e^{-\frac{1}{\chi}}\sum_{k=2}^{\infty}\left(\frac{1}{\chi}\right)^k (k\chi-1)\frac{\log k!}{k!}}_{H(\Omega|\vec{\mathrm{S}},\mathrm{T})/M\text{: average per-token order-uncertainty}}\right)}$$

where $\mathcal{R}_{P+T} \leq \mathcal{C}_{P+T}$.

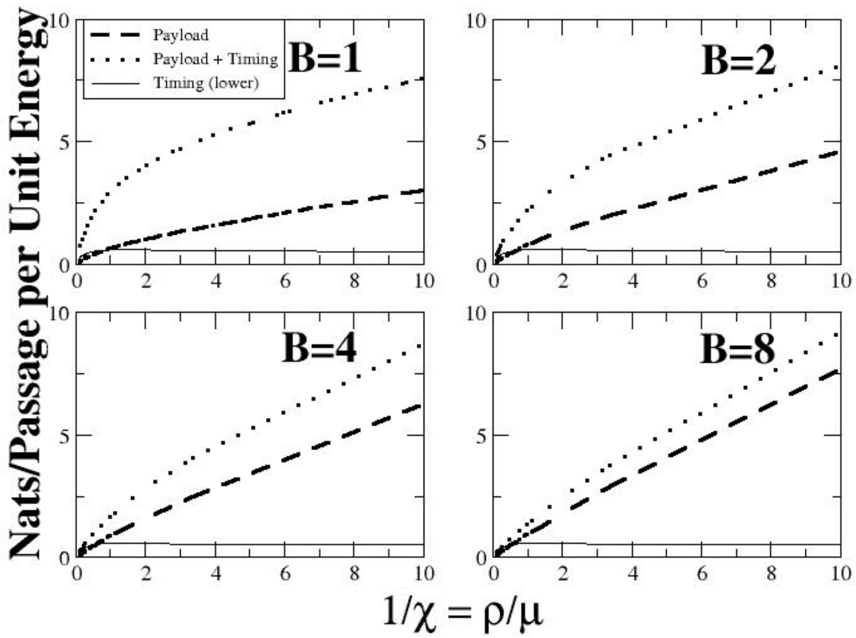
Info per Unit Energy



$\chi \leftrightarrow$ **passage rate per launch rate**

$c_0 = 1$, $c_1 = 0$, $\Delta c_1 = 1$

Info per Passage per Unit Energy



$\frac{1}{\chi} \leftrightarrow$ **launch rate per passage rate**

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And Now ....

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Numerical Play Time

Play Time Setup

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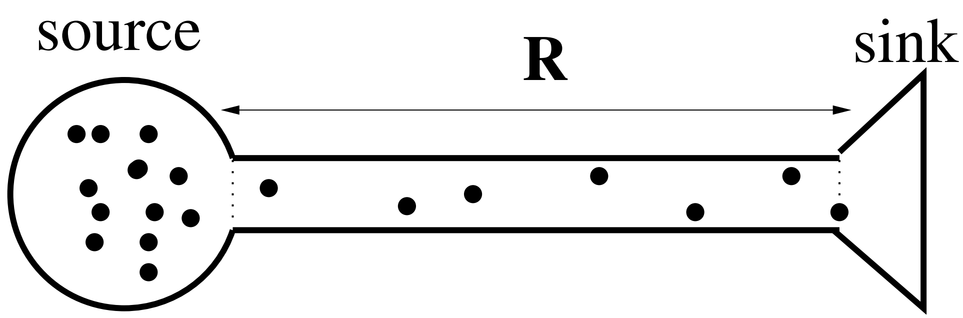

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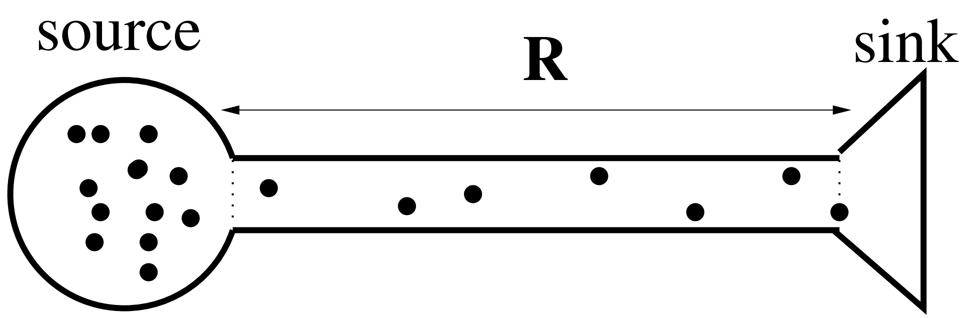
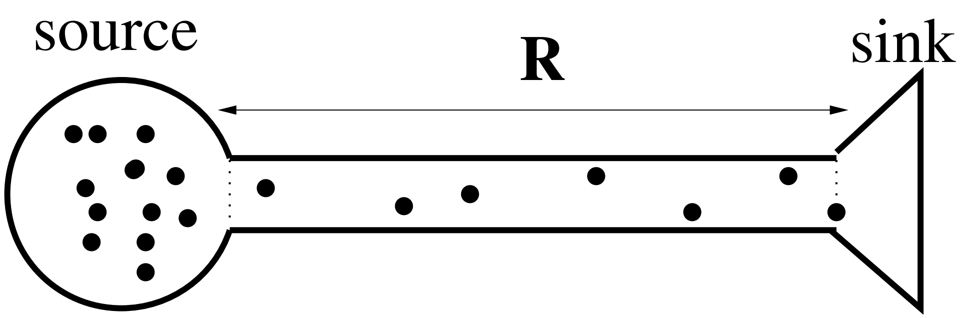


"Binary Protein" Token Construction $4B$**ATP** $= 3.2B \times 10^{-19} J$

Play Time Setup



"Binary Protein" Token Construction $4B$**ATP** $= 3.2B \times 10^{-19}J$

Diffusion Coefficient, $\mathcal{D}$ **in air:** $\approx 10^{-5}m^2/s$

Mean First Passage Time, $E[D] \approx \frac{R^2}{2\mathcal{D}}$

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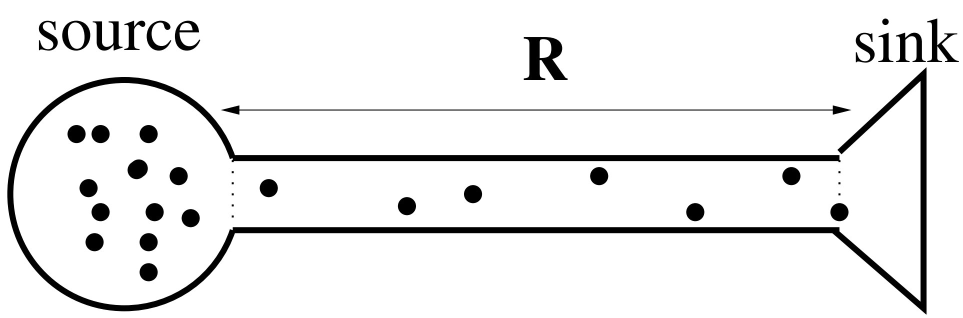

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Across a table ($1m$**):** $E[D] \approx 14hrs$ (need fan ☺)

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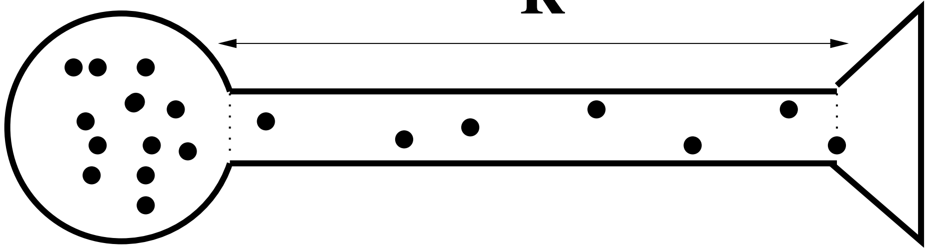

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Across a 0.1mm gap: $E[D] = 0.5ms$

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Across a table: $\approx$ 2 bits/day ($\approx 7 \times 10^{-24}$ W)

Across a 0.1mm gap: $\approx$ 10kb/s (≈ 3.2 fW)

Play Time Numbers

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Across a table: $\approx$ 2Kb/day ($\approx 7 \times 10^{-21}$ W)
Across a 0.1mm gap: $\approx$ 10Mb/s (≈ 3.2 pW)

Play Time Numbers

$$\frac{1}{\chi} = \frac{\rho}{\mu} = 1$$ (w/ identical tokens)

Across a table: $\approx$ 2 bits/day ($\approx 7 \times 10^{-24}$ W)
Across a 0.1mm gap: $\approx$ 10kb/s (≈ 3.2 fW)

$$\frac{1}{\chi} = \frac{\rho}{\mu} = 1000$$ (w/ $B = 1000$-bit tokens)

Across a table: $\approx$ 2Kb/day ($\approx 7 \times 10^{-21}$ W)
Across a 0.1mm gap: $\approx$ 10Mb/s (≈ 3.2 pW)

fiber: (100Tb/s@0.2W) 5×10^{14}**bits/J**

molecule: $\approx 3 \times 10^{18}$**bits/J**

Appropriately Awed Response

Netflix/SensorNet Distribution Fantasy

Disk Farm Fantasy

Suppose token construction energy cost $\ll$ fan energy cost

Disk Farm Fantasy

Suppose token construction energy cost $\ll$ fan energy cost

$1\mu g$ RNA per second $\Rightarrow 3.6 \times 10^{15}$ bits/sec

Molecular Communication Summary

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Timing is THE MOST Fundamental Treatment

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Need Bit Efficiency?

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Slow release with timing &/or small payload

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Scary Efficiencies

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Slow release with timing &/or small payload

Need Rate Efficiency?

Fast release with payload + timing or large payload

Scary Efficiencies

(beware transport latency)

If You Only Remember One Slide

A truck filled with storage media, driven across town, is a very reliable high bit rate channel.

–Comm. Theory Collective Subconscious

BUT ...

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A swarm of timed gnats

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BUT ...

A swarm of timed gnats with backpacks

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BUT ...

A swarm of timed gnats
with backpacks
in a breeze

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BUT ...

A swarm of timed gnats
with backpacks
in a breeze
could be better.