Signaling with Identical Tokens: lower bounds with energy constraints

Christopher Rose *Fellow, IEEE* and I. Saira Mian

***Abstract*—As system sizes shrink to the nanoscale, the usual macroscopic methods of communication using electromagnetic and acoustic waves become increasingly difficult owing to, essentially, a mismatch between realizable antenna sizes and the propagation characteristics of the medium. Thus, at the scale of microns and below, communication methods which utilize molecular messengers become increasingly attractive, a notion supported by the ubiquity of molecular signaling in biological systems, usually with identical molecules. In a large portion of previous work, time-varying signal molecule/token concentration is used as the observable and various analyses performed. However, from an information-theoretic standpoint, concentration masks the underlying process which consists, fundamentally, of signal token emission, diffusion through some medium, and reception. In this paper we establish a lower bound on identical token signaling with energy constraints and thereby indirectly provide max-min bounds on concentration-based signaling rates.**

***Index Terms*—Diffusion channel capacity, molecular signaling, timing channel capacity**

I. Introduction

SCALE-APPROPRIATE signaling methods become important as systems shrink to the nanoscale. For systems with feature sizes of microns and smaller, electromagnetic and acoustic communication become increasingly inefficient since energy coupling from the transmitter to the medium and from the medium to the receiver becomes difficult at usable frequencies. Biological systems, with the benefit of lengthy evolutionary experimentation, seem to have arrived at a moderately common solution to this signaling problem – use of identical tokens which diffuse through some medium between sender and receiver.

A fair amount of work on nano-scale communications has focused on diffusion of signaling molecules and a large portion of this work has explicitly considered time-varying concentration profiles as the fundamental signal measurement [1]–[4]. While this is an excellent first approach, concentration is a collective property of the process and masks the underlying physics of signal token release by the transmitter and capture by the receiver. This observation begs the question of truly fundamental limits on the capacity of such channels.

In what follows we consider a basic abstraction of molecular signaling wherein identical signaling molecules (tokens) are released from a transmitter according to some transmission schedule and each molecule is perfectly captured at the receiver with some medium-modulated reception schedule [5]. Building on previous work [5]–[8], we define the asymptotic (many channel use) model more precisely and provide *general* lower bounds on identical tokens channel capacity. In addition, since the molecule release and capture process comprises the underlying physics of concentration-based analyses, in the limit of large numbers of molecules these results supply a max-min bound for channels which use time-varying concentration as the information-carrying signal.

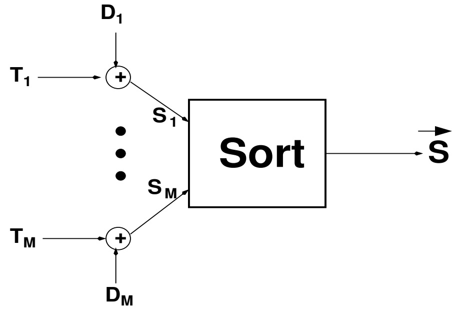

Fig. 1. Token release channel with reordering.

II. Problem Description

Repeating some of the development of [6]–[8] for clarity, assume that M identical tokens are emitted at times $\{T_m\}$, $m = 1, 2, ..., M$ and each is captured at the receiver. The duration of token m's passage between source and destination is a random variable D_m. These D_m are assumed i.i.d. with $f_{D_m}(d) = g(d) = G'(d)$ where $g()$ is some causal probability density with mean $\frac{1}{\mu}$ and CDF $G()$. We also assume that $g()$ contains no singularities. Thus, the first portion of the channel is modeled as a sum of random M-vectors

$$\mathbf{S} = \mathbf{T} + \mathbf{D} \tag{1}$$

for which we have

$$f_{\mathbf{S}}(\mathbf{s}) = \int_{\mathbf{0}}^{\mathbf{s}} f_{\mathbf{T}}(\mathbf{t})\mathbf{g}(\mathbf{s}-\mathbf{t})d\mathbf{t} \tag{2}$$

where $\mathbf{g}(\mathbf{s}-\mathbf{t}) = \prod_{m=1}^{M} g(s_m - t_m)$ and we impose an emission deadline, $T_m \leq \tau$, $\forall m \in \{1, 2, ..., M\}$. The associated emission time ensemble probability density $f_{\mathbf{T}}(\mathbf{t})$ is assumed causal, but otherwise arbitrary. We define the launch and capture of M tokens as a "channel use." If we assume multiple independent channel uses, then the usual coding theorems apply [9] and the channel's figure of merit is the mutual information between $\mathbf{T}$ and $\vec{\mathbf{S}}$, $I(\vec{\mathbf{S}}; \mathbf{T})$.

At this point it is tempting make a direct analogy to *Bits Through Queues* [10]. However, since the tokens are identical we cannot necessarily determine which arrival corresponds to which emission time. Thus, the final output of the channel is a reordering of the $\{s_m\}$ to obtain a set $\{\vec{s}_m\}$ where $\vec{s}_m \leq \vec{s}_{m+1}$, $m = 1, 2, ..., M-1$. We write this relationship as

$$\vec{\mathbf{S}} = P_{\Omega}(\mathbf{S}) \tag{3}$$

where $P_k()$, $k = 1, 2, \cdots, M!$, is a permutation operator and Ω is a permutation index which produces an arrival-time ordered $\vec{\mathbf{S}}$ from the argument $\mathbf{S}$. Incidentally, we define $P_1()$ as the identity permutation operator, $P_1(\mathbf{s}) = \mathbf{s}$. We note that the event $S_i = S_j$ $(i \neq j)$ is of zero measure owing to the no-singularity assumption on $g()$, Thus, for analytic convenience we will assume that $f_{\mathbf{S}}(\mathbf{s}) = 0$ whenever two or more of the s_m are equal. This assumption also assures that the Ω which produces $\vec{\mathbf{S}}$ in equation (3) is unique.

Thus, the density $f_{\vec{\mathbf{S}}}(\vec{\mathbf{s}})$ can be found by "folding" the density $f_{\mathbf{S}}(\mathbf{s})$ about the hyperplanes described by one or more of the s_m equal until the resulting probability density is nonzero only on the region where $s_m < s_{m+1}$, $m = 1, 2, ..., M-1$. Analytically we have

$$f_{\vec{\mathbf{S}}}(\mathbf{s}) = \begin{cases} \sum_{n=1}^{M!} f_{\mathbf{S}}(P_n(\mathbf{s})) & s_1 < s_2 < \cdots < s_m \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

We can likewise describe $f_{\vec{\mathbf{S}}|\mathbf{T}}(\mathbf{s}|\mathbf{t})$ as

$$f_{\vec{\mathbf{S}}|\mathbf{T}}(\mathbf{s}|\mathbf{t}) = \begin{cases} \sum_{n=1}^{M!} \mathbf{g}(P_n(\mathbf{s}) - \mathbf{t})\mathbf{u}(P_n(\mathbf{s}) - \mathbf{t}) & \text{ordered } s_i \\ 0 & \text{o.w.} \end{cases} \quad (5)$$

where $\mathbf{u}(P_n(\mathbf{s}) - \mathbf{t}) = \prod_{m=1}^{M} u([P_n(\mathbf{s})]_m - t_m)$ and $u()$ is the usual unit step function.

With these preliminaries done, we can now begin to examine the mutual information between $\mathbf{T}$, $\mathbf{S}$ and $\vec{\mathbf{S}}$.

III. Mutual Information Between $\mathbf{T}$ and $\vec{\mathbf{S}}$

The mutual information between $\mathbf{T}$ and $\mathbf{S}$ is

$$I(\mathbf{S}; \mathbf{T}) = h(\mathbf{S}) - h(\mathbf{S}|\mathbf{T}) \quad (6)$$

Since the S_i given the T_i are mutually independent, $h(\mathbf{S}|\mathbf{T})$ does not depend on $f_{\mathbf{T}}(\mathbf{t})$. Thus, maximization of equation (6) is simply a maximization of the marginal $h(S)$ over the marginal $f_T(t)$, a problem explicitly considered and solved for a mean T_m constraint in [10] and under a deadline constraint with exponential i.i.d. $\{D_m\}$ in [6].

The corresponding expression for the mutual information between $\mathbf{T}$ and $\vec{\mathbf{S}}$ is

$$I(\vec{\mathbf{S}}; \mathbf{T}) = h(\vec{\mathbf{S}}) - h(\vec{\mathbf{S}}|\mathbf{T}) \quad (7)$$

Unfortunately, $h(\vec{\mathbf{S}}|\mathbf{T})$ now *does* depend on the input distribution and the optimal form of $h(\vec{\mathbf{S}})$ is non-obvious. So, rather than attempting a brute force optimization of equation (7) by deriving order distributions [5], we first invoke simplifying symmetries.

Consider that an emission vector $\mathbf{t}$ and any of its permutations $P_n(\mathbf{t})$ produce statistically identical outputs $\vec{\mathbf{S}}$ owing to the reordering operation. Thus, any $f_{\mathbf{T}}()$ which optimizes equation (7) can be "balanced" to form an optimizing input distribution which obeys

$$f_{\mathbf{T}}(\mathbf{t}) = f_{\mathbf{T}}(P_n(\mathbf{t})) \quad (8)$$

for $n = 1, 2, ..., M!$ and $P_n()$ the previously defined permutation operator. We will therefore restrict our search to "hyper-symmetric" densities $f_{\mathbf{T}}(\mathbf{t})$ as defined by equation (8).

If we assume $f_{\mathbf{T}}()$ is hyper-symmetric, then it is easy to show that $f_{\mathbf{S}}()$ must also be hyper-symmetric. From equation (2) we have

$$f_{\mathbf{S}}(P_n(\mathbf{s})) = \int_{\mathbf{0}}^{P_n(\mathbf{s})} f_{\mathbf{T}}(\mathbf{t})\mathbf{g}(P_n(\mathbf{s}) - \mathbf{t})d\mathbf{t}$$

If we define $\mathbf{t}' = P_n^{-1}(\mathbf{t})$ we then have $f_{\mathbf{S}}(P_n(\mathbf{s})) = f_{\mathbf{S}}(\mathbf{s})$.

The hyper-symmetry of $f_{\mathbf{S}}(\mathbf{s})$ leads to a simple expression for $f_{\vec{\mathbf{S}}}(\mathbf{s})$. First we define $\mathcal{S}_1$ as the region in s-space for which $s_1 < s_2 < \cdots < s_m$. Similarly define disjoint regions $\mathcal{S}_n$ as those for which if $\mathbf{s} \in \mathcal{S}_n$ then $P_n(\mathbf{s}) \in \mathcal{S}_1$. That is, $\mathcal{S}_n$ is the region in s-space in which application of permutation operator $P_n()$ orders the components from smallest to largest.

Following equation (4) we have $f_{\vec{\mathbf{S}}}(\mathbf{s}) = M! f_{\mathbf{S}}(\mathbf{s})$ for $\mathbf{s} \in \mathcal{S}_1$. We can then write

$$\begin{aligned} h(\vec{\mathbf{S}}) &= -\textstyle\int_{\mathcal{S}_1} M! f_{\mathbf{S}}(\mathbf{s}) \log\left(M! f_{\mathbf{S}}(\mathbf{s})\right) d\mathbf{s} \\ &= -M! \textstyle\int_{\mathcal{S}_1} f_{\mathbf{S}}(\mathbf{s}) \log f_{\mathbf{S}}(\mathbf{s}) d\mathbf{s} - \log M! \end{aligned}$$

But since $f_{\mathbf{S}}(\mathbf{s})$ is hyper-symmetric, we also have

$$h(\vec{\mathbf{S}}) = -\sum_{n=1}^{M!} \int_{\mathcal{S}_n} f_{\mathbf{S}}(P_n(\mathbf{s})) \log f_{\mathbf{S}}(P_n(\mathbf{s})) d\mathbf{s} - \log M!$$

which owing to hypersymmetry becomes

$$h(\vec{\mathbf{S}}) = h(\mathbf{S}) - \log M! \quad (9)$$

We state this result as a theorem.

Theorem 1: If $f_{\mathbf{T}}()$ is a hyper-symmetric probability density function on emission times $\{T_m\}$, $m = 1, 2, .., M$, and the first passage density is non-singular, then the entropy of the size-ordered outputs $\vec{\mathbf{S}}$ is $h(\vec{\mathbf{S}}) = h(\mathbf{S}) - \log M!$

Next we turn to $h(\vec{\mathbf{S}}|\mathbf{T})$. A zero-measure edge-folding argument on the conditional density is not easily applicable here, so we resort to some sleight of hand. As before we define Ω as the permutation index number that produces an ordered output from $\mathbf{S}$. That is, $P_\Omega(\mathbf{S}) = \vec{\mathbf{S}} \in \mathcal{S}_1$. Specification of the random tuple $(\Omega, \vec{\mathbf{S}})$ is equivalent to specifying $\mathbf{S}$ and *vice versa*. Just as in our derivation of $h(\vec{\mathbf{S}})$, this equivalence requires that we exclude the zero-measure "edges" and "corners" of the density where two or more of the $\vec{\mathbf{s}}_i$ are equal.

We then have,

$$h(\mathbf{S}|\mathbf{T}) = h(\Omega, \vec{\mathbf{S}}|\mathbf{T}) = h(\vec{\mathbf{S}}|\mathbf{T}) + H(\Omega|\vec{\mathbf{S}}, \mathbf{T}) \quad (10)$$

which also serves as a definition for the entropy of a joint mixed distribution (Ω is discrete while $\vec{S}$ is continuous). We then rearrange equation (10) as

$$h(\vec{\mathbf{S}}|\mathbf{T}) = h(\mathbf{S}|\mathbf{T}) - H(\Omega|\vec{\mathbf{S}}, \mathbf{T}) \quad (11)$$

$H(\Omega|\vec{\mathbf{S}}, \mathbf{T})$ is the uncertainty about which S_m corresponds to which $\vec{S}_m$ given both $\mathbf{T}$ and $\vec{\mathbf{S}}$, and we note that $0 \leq H(\Omega|\vec{\mathbf{S}}, \mathbf{T}) \leq \log M!$ with equality on the right for any density where all the T_m are equal.

We can then, after assuming that $f_{\mathbf{T}}()$ is hyper-symmetric, write the ordered mutual information as

Theorem 2:

$$I(\vec{\mathbf{S}}; \mathbf{T}) = I(\mathbf{S}; \mathbf{T}) - \left(\log M! - H(\Omega|\vec{\mathbf{S}}, \mathbf{T})\right) \quad (12)$$

That is, an information degradation of size $\log M! - H(\Omega|\vec{\mathbf{S}}, \mathbf{T}) \geq 0$ is introduced by the sorting operation.

Since $h(\mathbf{S}|\mathbf{T})$ is a constant with respect to $f_{\mathbf{T}}(\mathbf{t})$, maximization of mutual the information in equation (12) requires we maximize $h(\mathbf{S}) + H(\Omega|\vec{\mathbf{S}}, \mathbf{T})$.

Mutual information is convex in $f_{\mathbf{T}}(\mathbf{t})$ and the space $\mathcal{F}_{\mathbf{T}}$ of feasible hyper-symmetric $f_{\mathbf{T}}(\mathbf{t})$ is clearly convex. Thus, we can in principle apply variational [11] techniques to find that hyper-symmetric $f_{\mathbf{T}}()$ which attains the unique maximum of equation (7). However, in practice, direct application of this method can lead to grossly infeasible $f_{\mathbf{T}}()$, implying that the optimizing $f_{\mathbf{T}}()$ lies along some "edge" or in some "corner" of the convex search space. In other work [7], [8] we consider the (somewhat difficult) problem of developing tight upper bounds on $I(\vec{\mathbf{S}};\mathbf{T})$. Here we concentrate on general lower bounds for this mutual information.

IV. Formalizing The Signaling Model

In the introduction we defined a channel use as the launch and capture of M tokens under a deadline constraint on emission times. We then (essentially) assumed sequential (or parallel) independent channel uses so that the figure of merit was the mutual information $I(\vec{\mathbf{S}};\mathbf{T})$. Here we consider more physically plausible conditions.

For instance, energy is a key resource in most systems. Thus, a good figure of merit for communication efficiency is nats/joule. In a biological context, a natural definition of capacity would then be nats/token since signal molecule construction (often a protein in biological systems) requires a known amount of energy. At roughly 4 ATP per amino acid [12], construction of a 100-amino acid protein would require 400 ATP – a significant cost even in comparison to an elevated 6×10^4 ATP/sec total energy budget during cell replication (E. Coli [13]) when one considers that many signaling molecules must be produced. In a human-engineered system, one could use pre-fabricated tokens, but even then, steady state operation would require tokens transport back to the receiver with some energy per token requirement. Thus, it seems useful to rewrite emission time constraints as a constraint on average token production ρ (tokens/second). Our previous emission constraint is then

$$\tau = \tau(M) = \frac{M}{\rho} \tag{13}$$

So, consider figure 2 where sequential transmissions of M tokens – channel uses – are depicted. We will assume a "guard interval" of some duration $\gamma(M,\epsilon)$ between successive transmissions so that all M transmissions are received before the beginning of the next channel use with high probability $(1-\epsilon)$. We further require that the average emission rate, $M/(\tau(M)+\gamma(M,\epsilon))$ satisfies

$$\lim_{\epsilon\to 0}\lim_{M\to\infty}\frac{M}{\tau(M)+\gamma(M,\epsilon)} = \rho \tag{14}$$

A convenient choice of $\gamma(M,\epsilon)$ is $\epsilon\tau(M)$ for any $\epsilon > 0$. We then require that

$$\lim_{M\to\infty}\text{Prob}\{\vec{S}_M \le \tau(M)(1+\epsilon)\} = 1 \tag{15}$$

We can interpret equation (15) as given arbitrarily small ϵ we can always find a finite M^* such that

$$\text{Prob}\{\vec{S}_M \le \tau(M)(1+\epsilon)\} > 1-\epsilon$$

$\forall M \ge M^*$. We can now derive conditions on first passage time densities under which equation (15) is true.

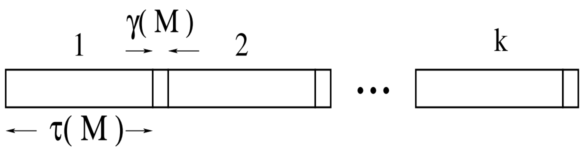

Fig. 2. k successive M-token emissions.

Calculating a CDF for $\vec{S}_M$ is in general difficult since emission times T_m might be correlated. However, for a fixed emission interval $[0,\tau(M)]$ we can readily calculate a worst case CDF for $\vec{\mathbf{S}}_M$ and thence a deterministic upper bound on the actual signaling epoch duration that is satisfied with probability $1-\epsilon$. That is, for a given emission schedule $\mathbf{t}$, the CDF for the final arrival is

$$F_{\vec{S}_M|\mathbf{t}}(s|\mathbf{t}) = \prod_{m=1}^{M} G(s-t_m)u(s-t_m)$$

so that

$$F_{\vec{S}_M}(s) = \int_{\mathbf{0}}^{\tau(\mathbf{M})} f_{\mathbf{T}}(\mathbf{t}) \prod_{m=1}^{M} G(s-t_m)u(s-t_m)d\mathbf{t}$$

However, it is easy to see that

$$F_{\vec{S}_M}(s) \ge G^M(s-\tau(M))u(s-\tau(M))$$

since $G(s-t_m)$ is monotone decreasing in t_m.

For $s = \tau(M)(1+\epsilon)$ we have

$$F_{\vec{S}_M}(\tau(M)(1+\epsilon)) \ge G^M(\frac{M\epsilon}{\rho}) \tag{16}$$

and we require $\lim_{M\to\infty} G^M(\frac{M\epsilon}{\rho}) = 1$ which for convenience, we rewrite as

$$\lim_{M\to\infty} M\log G(\frac{M\epsilon}{\rho}) = 0 \tag{17}$$

Thus, to satisfy equation (17), $(\log G(\epsilon\frac{M}{\rho}))^{-1}$ must be asymptotically supralinear in M.

If rewrite $\log G(\frac{M\epsilon}{\rho})$ in terms of the CCDF $\bar{G}()$ and note that $\log(1-x) \approx -x$ for x small, we have

$$\bar{G}(\frac{M\epsilon}{\rho}) - \epsilon \le \log\left(1-\bar{G}(\frac{M\epsilon}{\rho})\right) \le \bar{G}(\frac{M\epsilon}{\rho}) + \epsilon$$

for sufficiently large M. Thus, a first passage distribution whose CCDF satisfies

$$\lim_{M\to\infty} M\bar{G}(\frac{M\epsilon}{\rho}) = 0 \tag{18}$$

will also allow satisfaction of equation (15) with $\tau(M) = \frac{M}{\rho}$ and $\gamma(M,\epsilon) = \epsilon\tau(M)$.

Since all first passage times are non-negative random variables, the mean first passage time is

$$E[D] = \int_0^\infty \bar{G}(x)dx \tag{19}$$

The integral exists **iff** $1/\bar{G}(x)$ is asymptotically supralinear in x. Thus, if the mean first passage time $E[D]$ exists, then equation (18) is satisfied. Finally, in the limit of vanishing ϵ we have

$$\lim_{\epsilon\to 0}\lim_{M\to\infty}\frac{M}{\tau(M)+\gamma(M,\epsilon)} = \lim_{\epsilon\to 0}\frac{\rho}{1+\epsilon} = \rho$$

as required by equation (14)

V. Capacity Lower Bounds

A. Capacity Lower Bound in Nats Per Token

The maximum mutual information between $\mathbf{T}$ and $\vec{\mathbf{S}}$ per token given M launched tokens with timing constraint $\tau(M) = M/\rho$ is

$$C_q(M) = \frac{1}{M}\max_{f_{\mathbf{T}}()} I(\vec{\mathbf{S}};\mathbf{T}) \tag{20}$$

We define the limiting capacity in nats per token as

$$C_q = \lim_{M\to\infty} C_q(M) \tag{21}$$

$C_q(M)$ will be monotone increasing in M since concatenation of two emission intervals with durations $\tau/2$ and $M/2$ tokens each is more constrained than a single interval of duration τ with M tokens.

We can derive a simple lower bound on $C_q(M)$ by noting that equation (12) and the definition of equation (20) with $\tau(M)$ produces

$$\begin{aligned} C_q(M) &= \max_{f_{\mathbf{T}}()} \left[I(\mathbf{S};\mathbf{T}) + H(\Omega|\vec{\mathbf{S}},\mathbf{T}) \right] - \log M! \\ &\geq \max_{f_{\mathbf{T}}()} I(\mathbf{S};\mathbf{T}) - \log M! \end{aligned} \tag{22}$$

because $0 \leq H(\Omega|\vec{\mathbf{S}},\mathbf{T}) \leq M!$.

From [6] we know that the univariate maximum $I(S;T)$ subject to $T \leq \tau$ and a mean first passage time μ^{-1} is also minimized when the mean first passage time density $g()$ is exponential with parameter μ. Following [7] we have for any finite M and a finite launch deadline $\tau(M)$,

$$\max_{f_{\mathbf{T}}()} I(\mathbf{S};\mathbf{T}) \geq \min_{g()} \max_{f_{\mathbf{T}}()} I(\mathbf{S};\mathbf{T}) = M \log\left(1 + \frac{\mu\tau(M)}{e}\right) \tag{23}$$

which means,

$$C_q(M) \geq \log\left(1 + \frac{\mu\tau(M)}{e}\right) - \frac{\log(M!)}{M} \tag{24}$$

for a launch deadline $\tau(M)$.

Using equation (13) and Stirling's approximation, $\log M! = M\log M - M + O(\log(M))$ we have the following sequence of simplifications

$$\frac{1}{M}\left(M\log\left(1+\frac{\mu}{\rho e}M\right) - \log M!\right)$$
$$\log\left(1+\frac{\mu}{\rho e}M\right) - \log M + 1 - \frac{1}{M}O(\log(M))$$
$$\log\left(\frac{e}{M} + \frac{\mu}{\rho}\right) - \frac{1}{M}O(\log(M))$$

Defining $\chi = \frac{\mu}{\rho}$, the ratio of the token uptake rate to the release rate, and then taking the limit as $M \to \infty$ we obtain

$$\lim_{M\to\infty} C_q(M) = \log\chi \tag{25}$$

We summarize the results with a theorem:

Theorem 3: Given an average rate of signaling token production ρ as defined in equation (14) and any i.i.d. first passage time distribution with mean μ^{-1}, the timing channel capacity $C_q(\chi)$ in nats per token obeys

$$C_q(\chi) \geq \max\{\log\chi, 0\} \tag{26}$$

where $\chi = \frac{\mu}{\rho}$

We emphasize that the Theorem (3) bound is *general* and applies to *any* first passage time density $g()$ with mean μ^{-1}.

B. Capacity Lower Bound in Nats Per Unit Time

The duration of a signaling epoch is $\tau(M)+\gamma(M,\epsilon)$. Thus, for a given number M of emissions per channel use we define the channel capacity in nats per unit time as

$$\begin{aligned} C_t(M) &= \max_{f_{\mathbf{T}}()} \frac{I(\vec{\mathbf{S}};\mathbf{T})}{\tau(M)+\gamma(M,\epsilon)} \\ &= C_q(M,\tau(M)) \frac{M}{\tau(M)+\gamma(M,\epsilon)} \end{aligned}$$

where the $C_q(M,\tau(M))$ explicitly denotes an emission interval of duration $\tau(M)$. However, since we define $\rho = \frac{M}{\tau(M)+\gamma(M,\epsilon)}$ we then have

$$C_t(M) = \rho C_q\left(M, M\left(\frac{1}{\rho} - \frac{\gamma(M,\epsilon)}{M}\right)\right) \tag{27}$$

For any given tuple (ρ, M, ϵ), a positive interval duration $\tau(M)$, such that all tokens are received with probability $1-\epsilon$ by the end of the signaling epoch, either exists or does not. So, assume that a valid $\tau(M)$ exists. We know from the previous section that $2C_q(M/2) \leq C_q(M)$. We also know that

$$C_q(M, \tau(M) - \alpha) \leq C_q(M, \tau(M))$$

for $\alpha > 0$ since increasing the allowable emission interval cannot decrease the maximum mutual information. We also know from the previous section that if $E[D]$ exists, then the guard interval duration, $\gamma(M,\epsilon)$ can be sublinear in M. So, if we set $\tau(M) = M/\rho$, then $C_q\left(M, M\left(\frac{1}{\rho} - \frac{\gamma(M,\epsilon)}{M}\right)\right)$ is an increasing function of M whose limit is C_q. We summarize with the following theorem:

Theorem 4: If $E[D]$ exists, then the capacity in nats per unit time of the **token release timing channel** *obeys*

$$C_t = \rho C_q \tag{28}$$

where C_q is defined in equation (21) and ρ is the average token emission rate.

C. Special Case Lower Bounds: exponential first passage

We have the following theorem from [8]:

Theorem 5: For exponential first passage with parameter μ and launch deadline constraint τ, the corresponding $I(\mathbf{S};\mathbf{T})$-maximizing launch density is

$$\begin{aligned} f_{T_m}(t) &= \frac{1}{e+\mu\tau}\delta(t) + \frac{\mu}{e+\mu\tau}[u(t) - u(t-\tau)] \\ &+ \frac{e-1}{e+\mu\tau}\delta(t-\tau) \end{aligned} \tag{29}$$

$m = 1, 2, ..., M$. We then have

$$\begin{aligned} H(\Omega|\vec{\mathbf{S}},\mathbf{T}) = &\ E_{K_1}[\log K_1!] \\ &+ E_{K_2}\left[\left(K2\frac{\mu\tau}{1-p_2} - \frac{\mu\tau M}{(1-p_2)(\mu\tau+e)}\right)\log K_2!\right] \end{aligned} \tag{30}$$

where K_1 and K_2 are a binomial random variables over M trials with success probabilities $p_1 = \frac{e}{e+\mu\tau}$ and $p_2 = \frac{1}{e+\mu\tau}$, respectively.

And since the associated maximized $I(\mathbf{S};\mathbf{T})$ is $M\log(1+\frac{\mu\tau}{e})$ [6] we then have the following Lemma:

Lemma 1: For exponential first passage with parameter μ, a launch deadline of τ and $f_{\mathbf{T}}()$ given by equation (29) we have $I(\vec{\mathbf{S}};\mathbf{T})$ as

$$\begin{aligned} &M\log(1+\frac{\mu\tau}{e}) - \log M! + E_{K_1}[\log K_1!] \\ &+E_{K_2}\left[\left(K2\frac{\mu\tau}{1-p_2} - \frac{\mu\tau M}{(1-p_2)(\mu\tau+e)}\right)\log K_2!\right] \end{aligned} \tag{31}$$

where K_1 and K_2 are a binomial random variables over M trials with success probabilities $p_1 = \frac{e}{e+\mu\tau}$ and $p_2 = \frac{1}{e+\mu\tau}$, respectively.

Given exponential first passage, Lemma 1 provides a lower bound on $I(\vec{\mathbf{S}};\mathbf{T})$ for a deadline launch constraint. We now examine $\lim_{M\to\infty} \frac{I(\vec{\mathbf{S}};\mathbf{T})}{M}$ where we assume the launch constraint is specified by $\tau(M) = \frac{M}{\rho}$ as in sections V-A and V-B. To begin, remember that $\mu\tau(M) = \frac{\mu}{\rho}M \equiv \chi M$

and then note that $\binom{M}{k}\left(\frac{1}{1+\chi M}\right)^k\left(1-\frac{1}{1+\chi M}\right)^{M-k}$ reduces to $\frac{M(M-1)\cdots(M-k+1)}{M^k k!}\left(\frac{1}{\frac{1}{M\chi}+1}\right)^M\left(\frac{1}{\chi}\right)^k$. A similar series of simplifications [8] leads to the following theorem:

Theorem 6: For exponential first passage and $\mathbf{T}\in[0,M/\rho]^M$*, the channel capacity in nats per token obeys*

$$C_q(\chi)\geq\log\chi+e^{-\frac{1}{\chi}}\sum_{k=2}^{\infty}\left(\frac{1}{\chi}\right)^k(k\chi-1)\frac{\log k!}{k!}\qquad(32)$$

VI. Discussion & Conclusion

We have described a basic model for a tokens timing channel wherein identical tokens are released and travel independently to a receiver with information conveyed by the timing of arrivals. We have derived general machinery for the analysis of such channels and provided lower bounds on channel capacity under the assumption that the mean first passage time between sender and receiver is finite. The lower bounds on capacity are on the order of a half nat per first passage time.

It is worth noting that free diffusion (Brownian motion) first passage times are **not** finite and thus not well-behaved from an information theoretic capacity standpoint. However, in any finite spatial-extent system, physical constraints on tokens motion enforce finite first passage. It is also noteworthy that by considering tokens in the limit of large M per signaling interval, our results in principle bridge the gap between token channel descriptions and signaling agent concentration-based descriptions. That is, though the signaling problem formulation is epoch-based (M tokens per emission period τ, and $\rho=M/\tau$ constant), with large M (and concomitantly large τ), the "instantaneous" concentrations of tokens within an emission period are not so constrained. Thus, the "codewords" used by a token channel look like time-varying concentrations as time is blurred. And since such blurring obscures the fine-grain temporal information contained in $\vec{\mathbf{S}}$, the lower bounds on token channel capacity provide max-min bounds for concentration-based channel capacity.

It is also worth noting that the general lower bound of $C_q=\log\chi$ is *exactly* the capacity derived in *Bits Through Queues* [10] for the exponential service min-max channel. Thus, potentially disordered arrivals confer an advantage via implicit increases of token launch-time freedom.

The question of tokens number vs. timing information is worth exploring briefly. Consider that instead of fixing the number of tokens per epoch, we might send different numbers of tokens in each epoch. We have shown that $C_q(M)$ is at least linear in M. In contrast, the maximum amount of information conveyed per epoch by the *number* of tokens is exactly $\log M$ – strongly *sub*-linear in M. The argument also applies to $C_t(M)$ since the guard interval is proportionately larger for small M (larger-M intervals are more temporally efficient and therefore higher rate). Thus, in terms of information transfer, timing information seems strongly preferred, at least asymptotically.

Our lower bounds on channel capacity in nats per token (equation (26) and equation (32)) and the corresponding bounds in nats per passage time (equation (28)) are shown in figure 3. Increasing χ increases the emission interval relative the mean first passage time and thereby increases the information content of any individual token. In addition, since successive tokens may be less likely to interchange position, $\frac{1}{M}\left(\log M!-H(\Omega|\vec{\mathbf{S}};\mathbf{T})\right)$ approaches zero. Thus, the simple lower bound of equation (26) (and correspondingly equation (28)) meets the lower bound for exponential first passage which has minmax $I(\mathbf{S};\mathbf{T})$. But perhaps most interesting is the implication that there may exist optimum emission rates for a given channel as evidenced by the shape of the C_t/μ curves in figure 3. This feature echos [1] where an optimum burst interval for signaling molecules in a diffusive channel was derived. However, since we do not yet know the channel capacity nor useful upper bounds, we do not know how tight our lower bounds are. It is therefore premature to say whether an optimum emission rate is a feature of the identical token timing channel, as tantalizing a prospect as that may be.

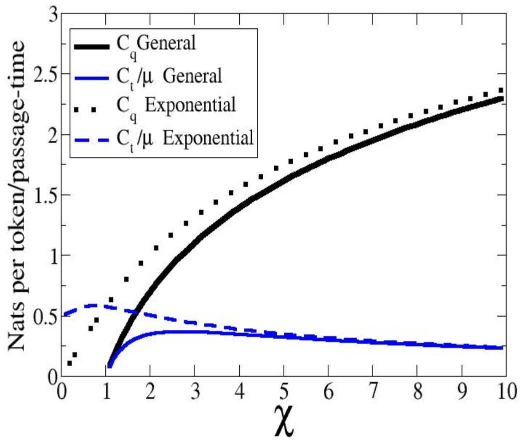

Fig. 3. Lower bounds for C_q, $\frac{C_t}{\mu}$ vs. χ.

References

[1] A. Einolghozati, M. Sardari, A. Beirami, and F. Fekri. Capacity of Discrete Molecular Diffusion Channels. In *IEEE International Symposium on Information Theory (ISIT) 2011*, pages 603–607, July 2011. ISBN: 978-1-4577-0594-6.

[2] Josep Miquel Jornet Ian F. Akyildiz and Massimiliano Pierobon. Nanonetworks: A New Frontier in Communications. *Communications of the ACM*, 54(11):84–89, 2011.

[3] M. Pierobon and I. Akyildiz. Capacity of a diffusion-based molecular communication system with channel memory and molecular noise. *Information Theory, IEEE Transactions on*, 2012. (to appear).

[4] I.F. Akyildiz, W. Su, Y Sankarasubramaniam, and E. Cayirci. A Survey on Sensor Networks. *IEEE Communication Magazine*, 40(8):102–116, 2002.

[5] A.W. Eckford. Nanoscale communication with Brownian motion. In *CISS'07*, pages 160–165, 2007. Baltimore.

[6] Y-L Tsai, C. Rose, R. Song, and I.S. Mian. An Additive Exponential Noise Channel with a Transmission Deadline. In *IEEE International Symposium on Information Theory (ISIT) 2011*, pages 598–602, July 2011. ISBN: 978-1-4577-0594-6.

[7] R. Song, C. Rose, Y-L Tsai, and I.S. Mian. Wireless Signaling with Identical Quanta. In *IEEE WCNC*, Paris, April 2012.

[8] C. Rose, R. Song, and I.S. Mian. Timing Channels with Multiple Identical Quanta. *IEEE Transactions on Information Theory*, 2013. (submitted – available: http://arxiv.org/abs/1208.1070).

[9] T.M. Cover and J.A. Thomas. *Elements of Information Theory*. Wiley-Interscience, 1991.

[10] V. Anantharam and S. Verdu. Bits Through Queues. *IEEE Transactions on Information Theory*, 42(1):4–18, January 1996.

[11] F.B. Hildebrand. *Advanced Calculus for Applications*. Prentice Hall, Englewood Cliffs, NJ, 1976.

[12] D.L. Nelson and M.M. Cox. *Lehninger Principles of Biochemistry*. Freeman, 2005. 4th Ed.

[13] A. Lehninger. *Biochemistry: the Molecular Basis of Cell Structure and Function*. Worth Publishing, 1975.

Acknowledgements

This work was supported by NSF-CDI-0835592. Dear anonymous reviewers: the figures are bigger (A4) and although $C_{\mathbf{q}}$ persists, quanta are now tokens – and that's a good thing. Thanks!