Communicating with Identical Tokens: lower bounds

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The Heroic Picture

What can a cell(s) tell the world?

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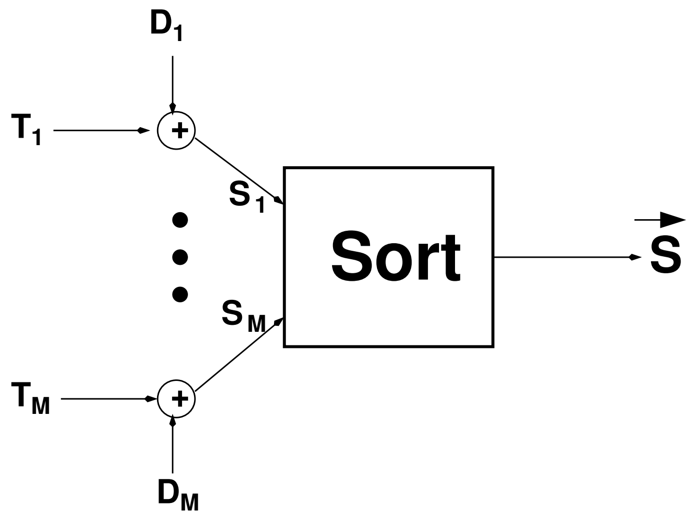
Use IT *bounds* to avoid modeling morass

What Can a Cell Tell the World: abstraction



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$$\vec{\mathbf{S}} = \mathrm{Sort}[\mathbf{S}]$$

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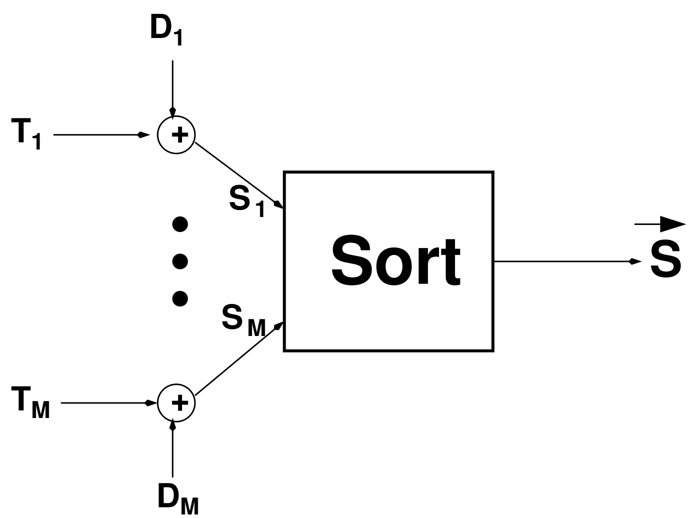

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Simplistic but fundamental model

Mutual Information

$$
\begin{aligned}
I(\mathbf{S};\mathbf{T}) &= h(\mathbf{S}) - h(\mathbf{S}|\mathbf{T}) \\
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Easy, right?

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$$I(\vec{\mathbf{S}};\mathbf{T}) = h(\vec{\mathbf{S}}) - h(\vec{\mathbf{S}}|\mathbf{T}) = \ ?$$

EGAD!!! (Chris FEARS Order Distributions)

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Consider Only Hypersymmetric $\mathbf{T}$

$$\max_{f_{\mathbf{T}}} I(\vec{\mathbf{S}}, \mathbf{T})$$

More Symmetry

$f_{\mathbf{T}}()$ hypersymmetry $\rightarrow$ $f_{\mathbf{S}}()$ hypersymmetry

$f_D()$ non-singular $\rightarrow$ $\mathbf{S}$ continuous

"Edges and Corners" of $f_{\mathbf{S}}()$ have **zero measure**

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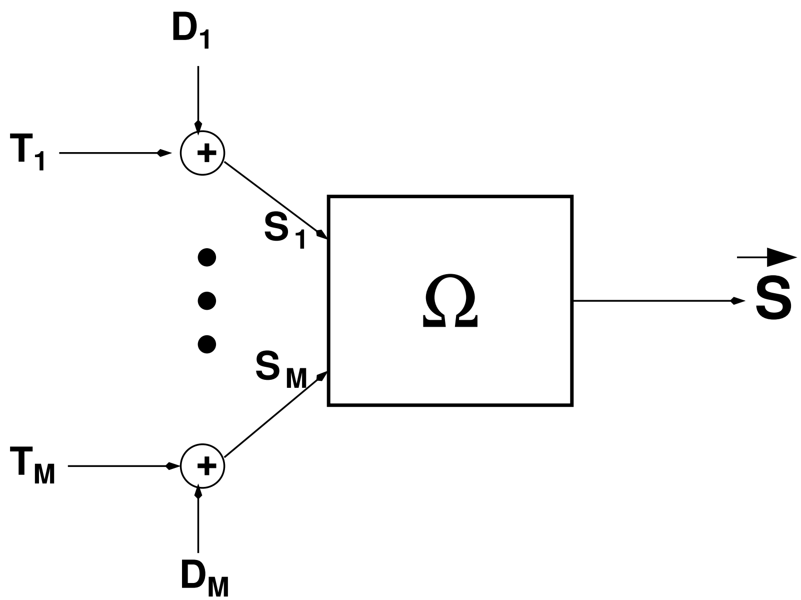
$$h(\vec{\mathbf{S}}) = h(\mathbf{S}) - \log M!$$

Channel Redux



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$$\boldsymbol{I(\vec{\mathrm{S}}; \mathrm{T}) = \underbrace{h(\mathrm{S}) + H(\Omega|\vec{\mathrm{S}}, \mathrm{T})}_{\text{The Money!}} - \underbrace{(\log M! + h(\mathrm{D}))}_{\text{constant}}}$$

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$H(\Omega|\vec{\mathbf{S}}, \mathbf{T})$ maximized by correlated $\mathbf{T}$

$$H(\Omega|\vec{\mathbf{S}}, \mathbf{T}) \leq \log M!$$

(i.e., identical launch times $T_1 = T_2 = \cdots = T_M$)

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$\exists$ closed form results for $H(\Omega|\vec{\mathbf{S}}, \mathbf{T})$

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(obvious data processing aside)

Cosmic Wimp-Out

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(throw out $H(\Omega|\vec{\mathbf{S}},\mathbf{T})$)

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$$\frac{\textbf{Energy}}{\textbf{Time}} = \frac{\textbf{Tokens}}{\textbf{Launch Epoch}} = \frac{M}{\tau(M)} = \rho$$

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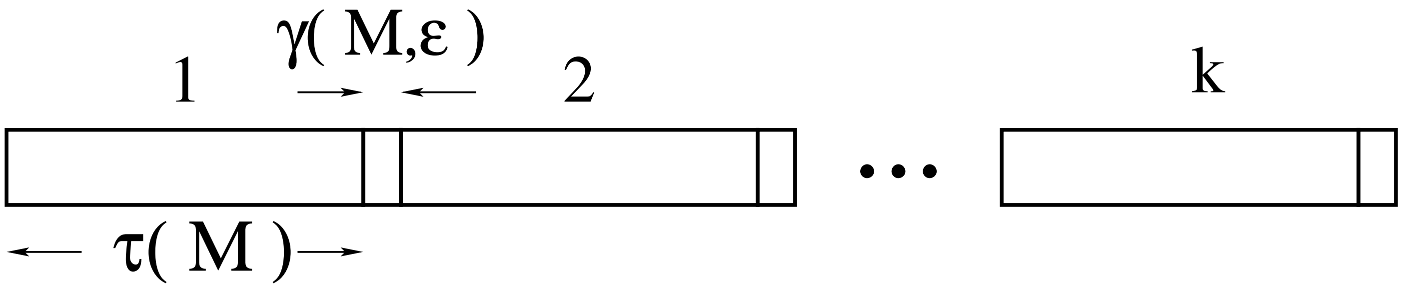

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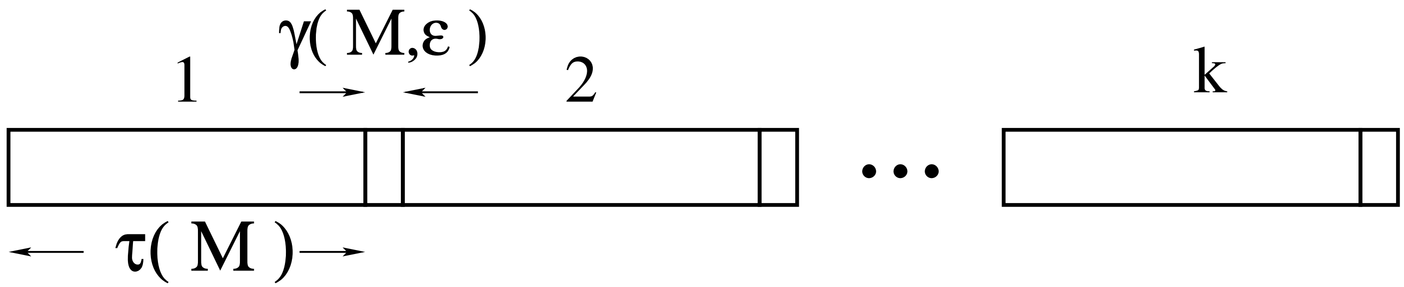

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Power Constraint:

$$\rho = \lim_{\epsilon \to 0} \lim_{M \to \infty} \frac{M}{\tau(M) + \gamma(M,\epsilon)}$$

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PUNCHLINE: all ok if $E[D]$ exists

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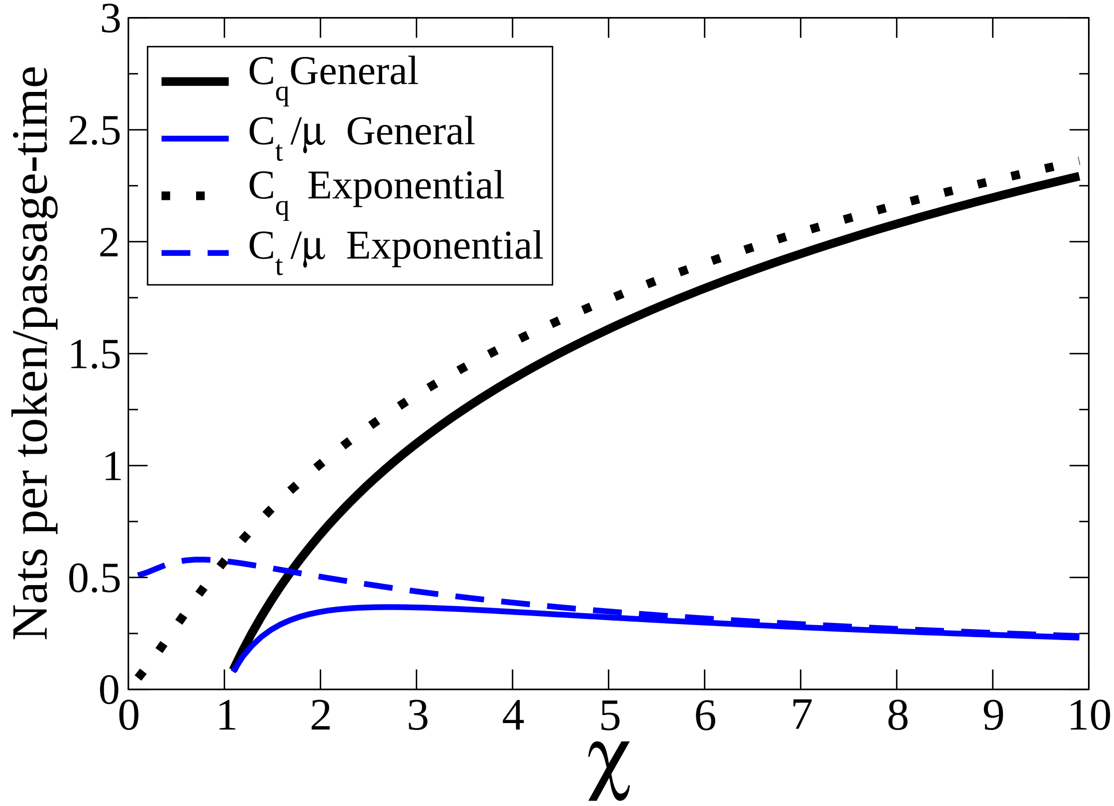
Completely General

Plot and Comparison with Exponential Special Case



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Really need a good upper bound