Communicating with Identical Tokens: upper bounds

Christopher Rose[1]
I. Saira Mian[2]

[1]Rutgers University, WINLAB
[2] University College London

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Molecular Communication

- **Eu/Prokaryotic systems intercommunicate**

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  - Emission/reception of special molecules (tokens)

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Intriguing science & engineering

LOTS of recent work in the area

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It's a Snake!
It's a Tree!
It's a Wall!
It's a Spear!
It's a Rope!

Chris Is Getting Old – and CRANKY!

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Is There a Unifying Elephant?

(framework)

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What can a cell tell the world?

(fundamental limits)

Cartoon and Desiderata

Diffusing Molecules Cartoon

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What Are We looking For?

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General multipurpose “outer bound” model

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Use IT *bounds* to avoid modeling morass

My Stab At A Unified Framework

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Timing Channel Is Fundamental!

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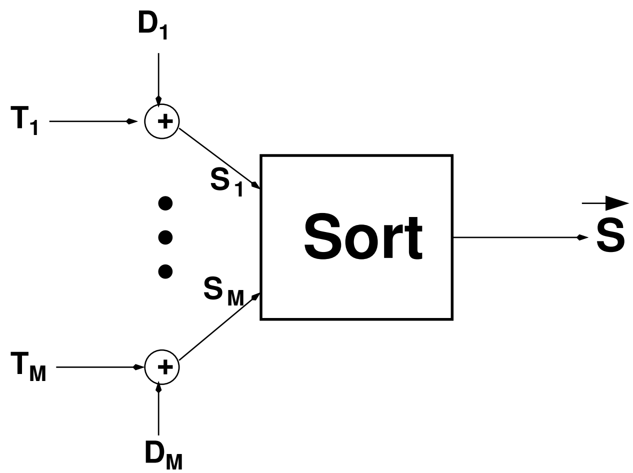
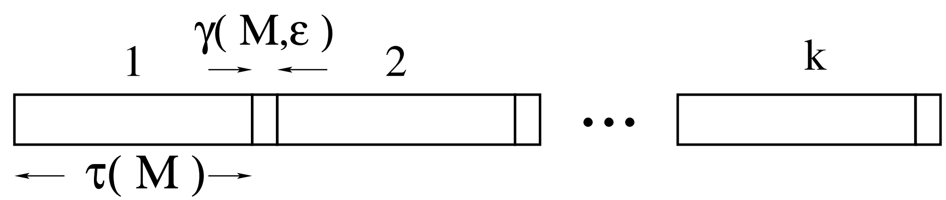
Channel abstraction
Information-theoretic modeling
Past Successes and Frustrations
Help From An Old Reliable Friend
Gyrations To An Upper Bound
Capacity Sandwich

Abstraction of "What Can a Cell Tell the World?"





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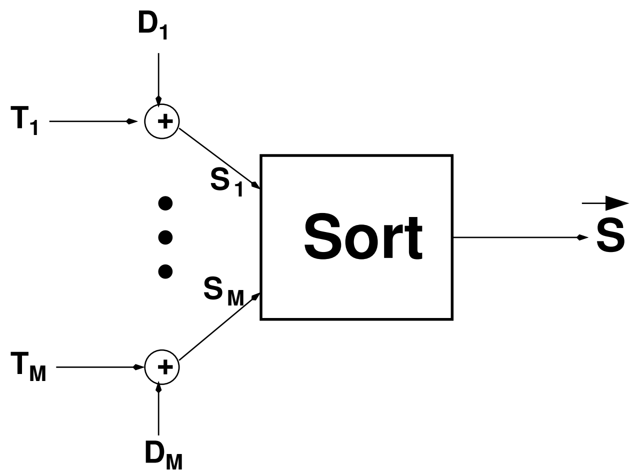

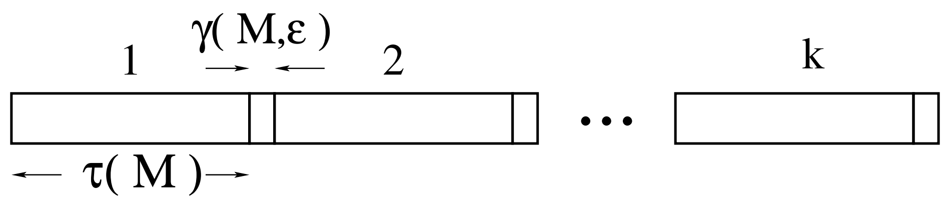

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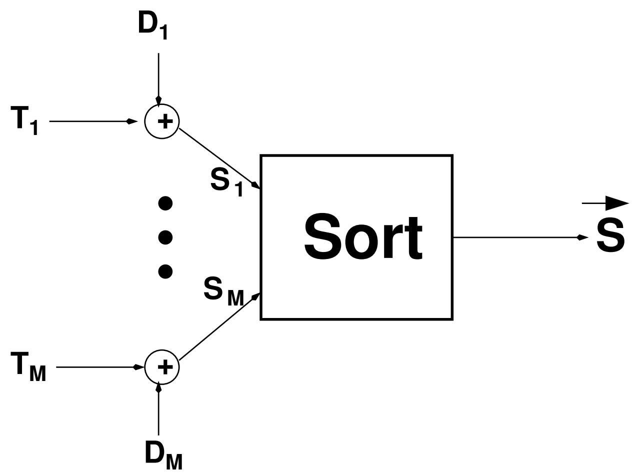

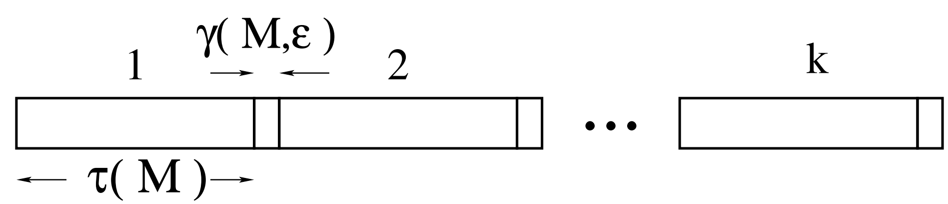

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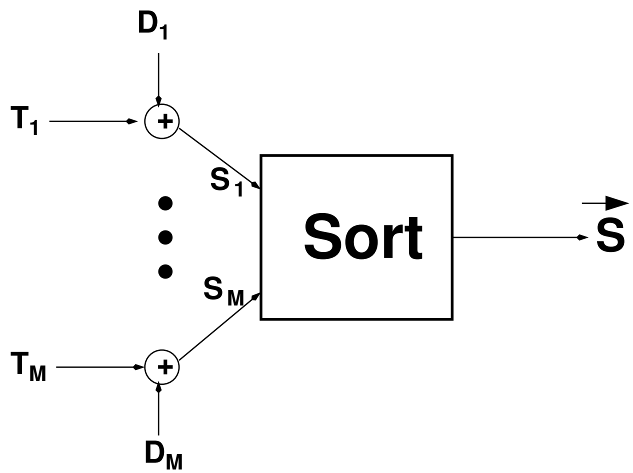

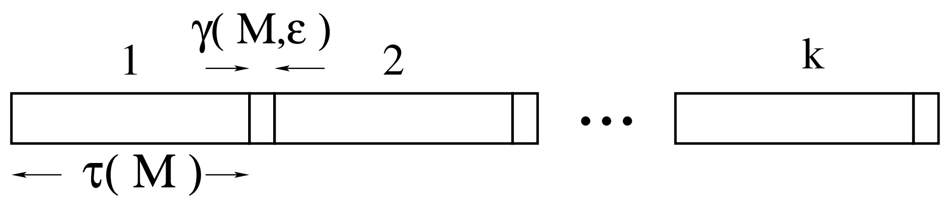

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First passage time: $E[D] = 1/\mu$

Tokens cost energy!!: $\rho \equiv M/\tau(M)$

Mutual Information

M tokens on an interval $\tau(M)$

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\begin{aligned}
I(\mathbf{S};\mathbf{T}) &= h(\mathbf{S}) - h(\mathbf{S}|\mathbf{T}) \\
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&\le M\left(h(S) - h(D)\right), \qquad \text{(i.i.d. } \mathbf{D}\text{)}
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$$I(\vec{\mathbf{S}};\mathbf{T}) = h(\vec{\mathbf{S}}) - h(\vec{\mathbf{S}}|\mathbf{T}) = \text{?}$$

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$$\exists \; M! \; \mathbf{T} \xrightarrow{\Omega} \vec{\mathbf{T}}$$

(permuatation operator $P_\Omega()$, index Ω)

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Consider Only Hypersymmetric $\mathbb{T}$

$$\max_{f_{\mathbf{T}}} I(\vec{\mathbf{S}}, \mathbf{T})$$

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$f_{\mathbf{T}}()$ hypersymmetry $\rightarrow$ $f_{\mathbf{S}}()$ hypersymmetry

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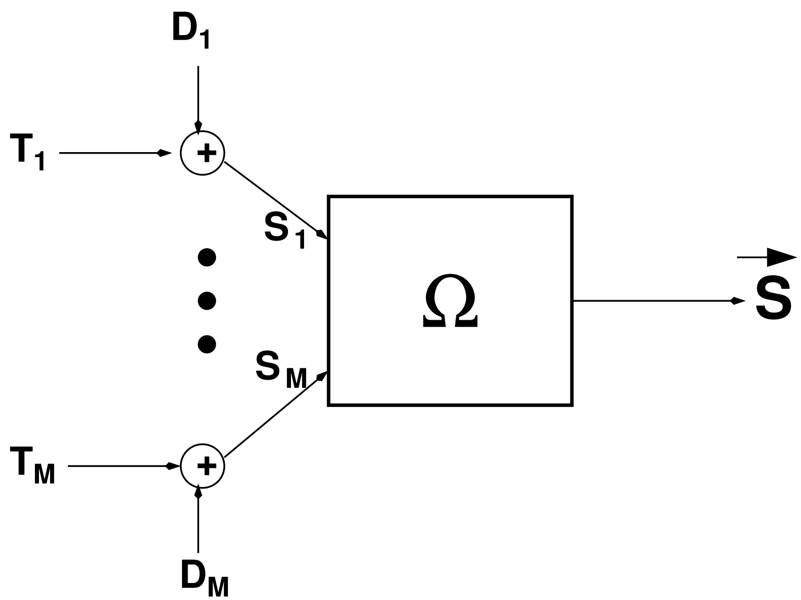
$$h(\vec{\mathbf{S}}) = h(\mathbf{S}) - \log M!$$

Channel Redux



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$$I(\vec{\mathbf{S}}; \mathbf{T}) = \underbrace{h(\mathbf{S}) + H(\Omega|\vec{\mathbf{S}}, \mathbf{T})}_{\text{The Money!}} - \underbrace{(\log M! + h(\mathbf{D}))}_{\text{constant}}$$

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$H(\Omega|\vec{\mathbf{S}}, \mathbf{T})$ maximized by correlated $\mathbf{T}$

$$H(\Omega|\vec{\mathbf{S}}, \mathbf{T}) = \log M!$$

identical launch times $T_1 = T_2 = \cdots = T_M$

My Past Personal Struggles

$\exists$ closed form results for $H(\Omega|\vec{\mathbf{S}}, \mathbf{T})$

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$$\max_{f_{\mathbf{T}}()} h(\mathbf{S}) + H(\Omega|\vec{\mathbf{S}}, \mathbf{T}) \geq ?$$ (Last Year's Talk)

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This Year's Struggle

$$\max h(\mathbf{S}) + H(\Omega|\vec{\mathbf{S}}, \mathbf{T}) \leq \textbf{?}$$

(obvious data processing aside)

Hello Jensen My Old Friend

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$$H(\Omega|\vec{\mathbf{S}},\vec{\mathbf{t}}) \leq H^{\uparrow}(\vec{\mathbf{t}}) = \sum_{\ell=1}^{M-1} \log(1+\ell) \sum_{m=\ell}^{M-1} \sum_{|\bar{\mathbf{x}}|=\ell} \prod_{j=1}^{m} \bar{G}^{\bar{x}_j}(\vec{t}_{m+1}-\vec{t}_j) G^{1-\bar{x}_j}(\vec{t}_{m+1}-\vec{t}_j)$$

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Theorem 1.

$$H(\Omega|\vec{\mathbf{S}},\mathbf{T}) \le E_{\mathbf{T}}\left[H^{\uparrow}(\mathbf{T})\right] \le M \log\left(1 + \frac{M-1}{2}\gamma_T\right)$$

where

$$Q(\cdot) \equiv \bar{G}(|\cdot|) \quad \& \quad \gamma_T = E_{\mathbf{T}}\left[Q(T_1 - T_2)\right]$$

Maximize $h(\mathrm{S})$ (Euler-Lagrange)

$$\arg \max_{\{f_{\mathbf{S}}(),\text{fixed } \gamma_S\}} h(\mathbf{S}) = \frac{1}{A(\beta)} e^{\beta \sum_{\substack{i,j \\ i \neq j}} Q(s_i - s_j)}$$

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with β chosen to satisfy $E[Q(S_1 - S_2)] = \gamma_S$.

Bounding $H(\Omega|\vec{\mathbf{S}}, \mathbf{T})$ (for exponential first passage)

Theorem 2. *If the first passage density $f_D()$ is exponential then*

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Adaptable to other first passage densities(!)

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yields

$$\Rightarrow \gamma_S^* = \gamma_S(\beta^*) = \frac{1-\beta^*}{(M-1)\beta^*}$$

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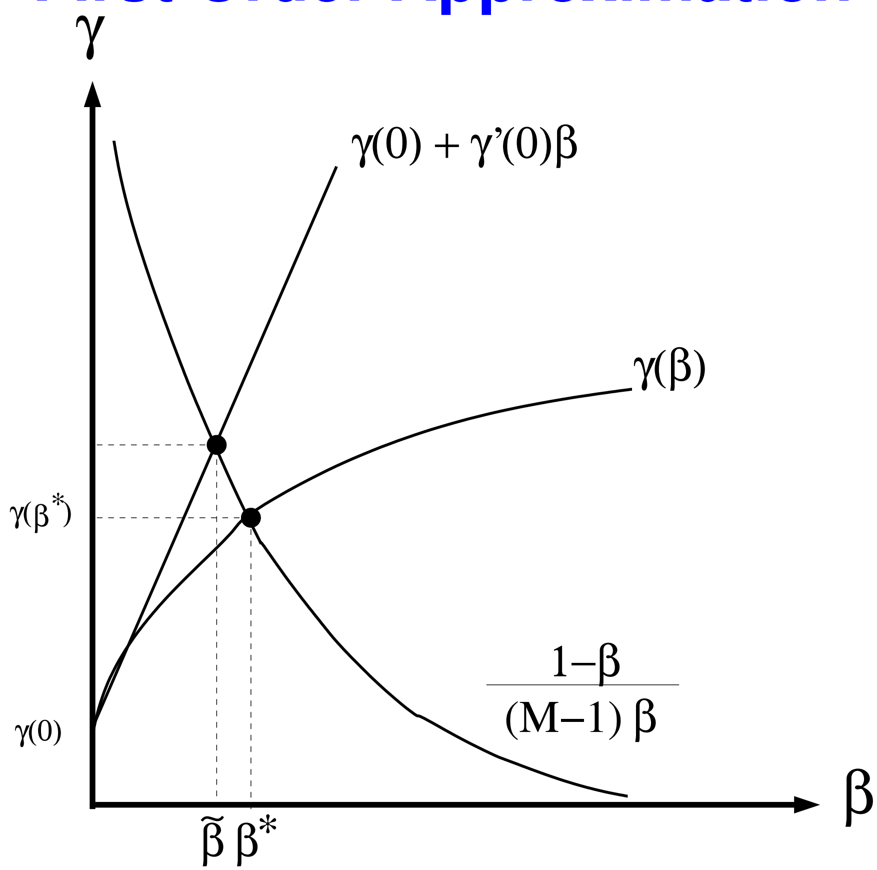
But $\gamma_S(\beta)$ concave ...

First Order Approximation



$$\gamma_S(\beta) \leq \gamma_S(0) + \gamma'_S(0)\beta$$

Capacity Upper Bounds

Theorem 3. *If the first passage density $f_D()$ is exponential with parameter μ and the average rate at which tokens are released is ρ, and we define $\chi = \mu/\rho$, then the capacity per token, C_m is upper bounded by*

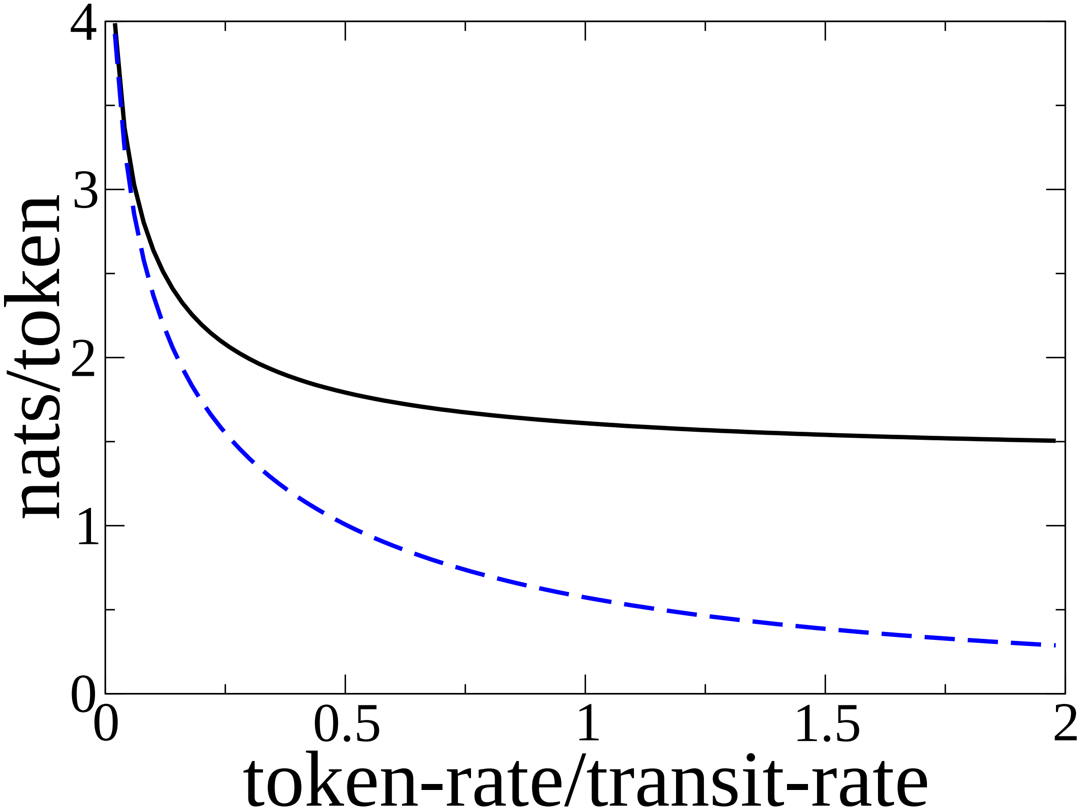
$$C_m \leq \log\left(\chi + \tilde{\beta}\left(\frac{8}{\chi} + 2\right) + 2\right)$$

and the capacity per unit time is upper bounded by

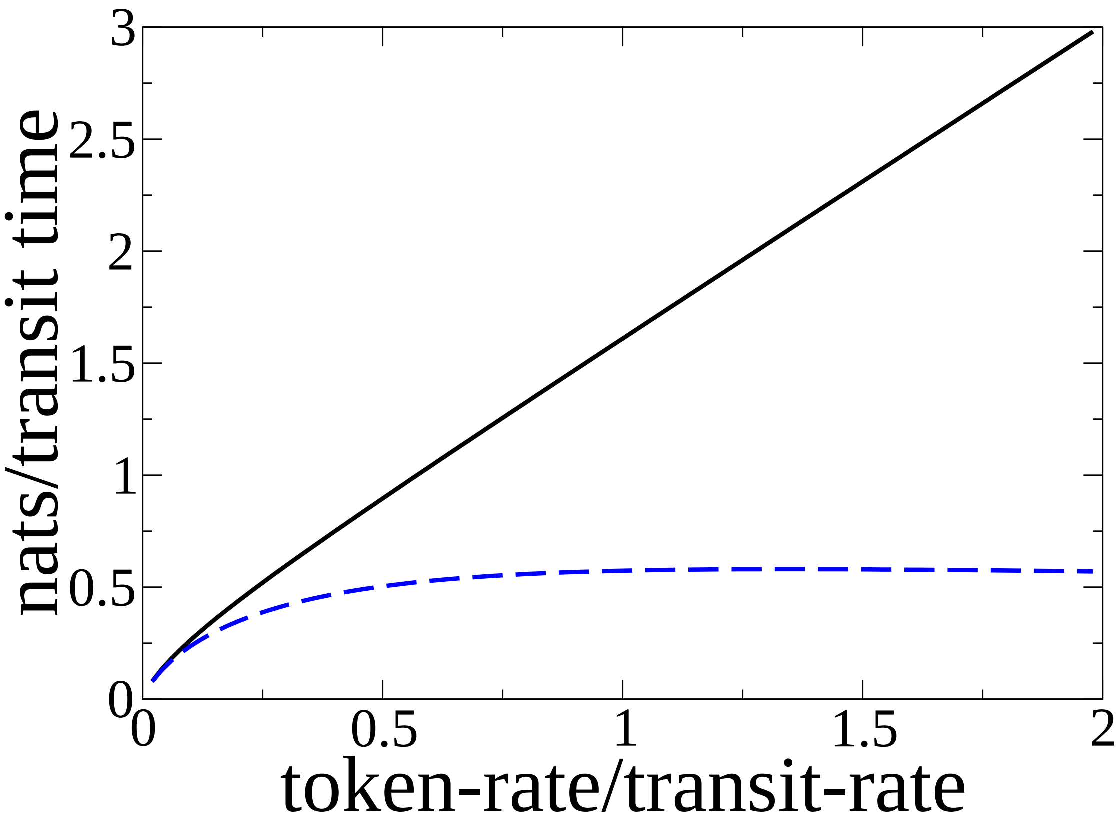
$$C_t \leq \rho \log\left(\chi + \tilde{\beta}\left(\frac{8}{\chi} + 2\right) + 2\right)$$

where $\tilde{\beta} = \frac{\chi^2\sqrt{1+\frac{12}{\chi}+\frac{36}{\chi^2}}-(\chi^2+2\chi)}{16+4\chi}$

C_m Sandwich: exponential special case



C_t/μ Sandwich: exponential special case



Summary + Next

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  - Large $\rho/\mu \Rightarrow$ **concentration signaling**
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- **News Flash:** Relationship to network coding? (stay tuned)