Mean Internodal Distance in Regular and Random Multihop Networks

Christopher Rose, *Member, IEEE*

Abstract—**The minimum necessary aggregate link capacity in a telecommunications network is directly proportional to the mean distance between nodes. The mean internodal distance is therefore an important network characteristic. This study provides the surprising result that most network topologies, including those constructed at random, display mean internodal distances caparable to those of many carefully designed networks. Thus, careful selection of network topology to minimize the mean internodal distance may be important in only the most sensitive applications. And even in such sensitive applications, an almost randomly chosen network topology may be the best choice.**

I. Introduction

MULTIHOP networks must pass messages between source and destination nodes via intermediate links and nodes. Assuming propagation delay is ignored, the mean amount of time the average message stays in a multihop network is directly proportional to the mean distance, in hops, between nodes. Thus, the aggregate link capacity of the network is also directly proportional to this "mean internodal distance." In addition the mean internodal distance is proportional to the average delay between sending and receiving a message as well as being inversely proportional to the channel efficiency of a network.[1]

This study considers the following question:

> *Given a set of N nodes and L directed links, what level of connectivity and mean internodal distance can be expected of most networks?*

Similar questions have previously been asked of completely random graphs [1], [2]. Unfortunately, primarily extremal properties such as the maximum internodal distance (graph diameter) rather than mean properties have been studied. An extremal property such as graph diameter, although useful in upper-bounding the minimum mean internodal distance, does not relate intimately to the important network attribute of aggregate capacity. Thus, diameter is only of limited usefulness in this regard.

Paper approved by the Editor for Communication Switching of the IEEE Communications Society. Manuscript received March 2, 1990; revised October 10, 1990.

The author is with the Department of Electrical and Computer Engineering, Rutgers University, Piscataway, NJ 08855.

IEEE Log Number 9201161.

[1]The channel efficiency η of a network is the proportion of new source traffic to old traffic *en route* to its destination. (see [4]). The mean internodal distance is the inverse of η. Thus, under uniform loading, only a completely connected network has a channel efficiency of 1.0 since its mean internodal distance is exactly one.

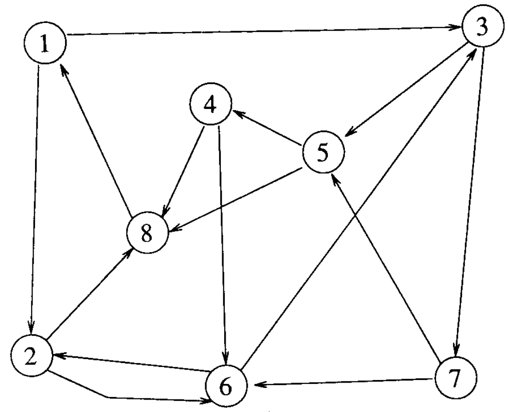


Fig. 1. Model network with 8 nodes and 15 directed links.

Through analysis and experimentation this study has revealed that many network topologies show remarkably similar characteristics. Thus, careful selection of network topologies *vis a vis* mean internodal distance may be warranted only in the most sensitive applications. A further surprising result of this study is that a large number of almost randomly constructed networks show smaller mean internodal distances than more regular seemingly efficient network topologies such as Shufflenet [4], [5], Hypercube [8] and others.

II. Background

A. Concepts and Definitions

- *The Connection Matrix*

A useful representation of network connectivity is the connection matrix.[2] The connection matrix, $\boldsymbol{C}$, is an $N \times N$ matrix of ones and zeroes in which a nonzero entry, c_{ij} corresponds to a link from node i to node j. The number of nonzero entries is therefore L.

- *In-Degree and Out-Degree*

Define the number of links emanating from node i as p_i^{out} and the number of links impinging on node i as p_i^{in}. The terms p_i^{out} and p_i^{in} are also called, respectively, the out-degree and in-degree of node i. For example, $p_1^{\text{out}} = 2$ and $p_1^{\text{in}} = 1$ for the network of Fig. 1.

Since all outgoing links must impinge on a node and all incoming links must emanate from a node, the average in-degree must equal the average out-degree. This average

[2]Similar to the *adjacency matrix* [3].

degree, $p \equiv L/N$ where L is the number of links in the network, is a network parameter of considerable interest since it is intimately related to the mean distance between nodes.

• *Mean Internodal Distance*

Consider the network depicted in Fig. 1. Its mean internodal distance, $\bar{h}$, may be calculated as follows. Starting at node 1,[3] nodes 2 and 3 may be reached in one hop. Nodes 5, 6, 7, and 8 may in turn be reached from nodes 2 and 3 in one hop. Thus 5, 6, 7, and 8 are reachable in two hops from node 1. Finally, nodes 1, 2, 3, 4, 5, 6, 7, and 8 are reachable from nodes 5, 6, 7, and 8. However, only node 4 has not been previously visited. Therefore only node 4 is called reachable in three hops.

To calculate the mean internodal distance first define h_{ik} as the number of nodes reachable in k hops starting at node i. Then define d_i as the number of hops necessary to reach the node(s) farthest from node i. Then, in the preceding example we have

$$\begin{aligned} h_{10} &= 1 \\ h_{11} &= 2 \\ h_{12} &= 4 \\ h_{13} &= 1 \\ d_1 &= 3. \end{aligned} \tag{1}$$

The mean distance from node 1 to the rest of the network is then

$$\bar{h}_1 = \frac{h_{11} + 2h_{12} + 3h_{13}}{\sum_{k=1}^{d_1} h_{1k}} = \frac{13}{8} = 1.625. \tag{2}$$

In general,

$$\bar{h}_i = \frac{\sum_{k=0}^{d_i} k h_{ik}}{\sum_{k=1}^{d_i} h_{ik}}. \tag{3}$$

Another measure which will prove useful is the mean number of new nodes reached on the kth hop, averaged over all network nodes,

$$h_k = \frac{1}{N} \sum_{i=1}^{N} h_{ik}. \tag{4}$$

The mean internodal distance for the network is then

$$\bar{h} = \frac{1}{N} \sum_{i=1}^{N} \bar{h}_i = \frac{\sum_{k=0}^{\infty} k h_k}{\sum_{k=0}^{\infty} h_k}. \tag{5}$$

For the network shown, $\bar{h} = 1.84375$. A more formal statement of this procedure is

$$\bar{h} = \frac{1}{N} \sum_{i=1}^{N} \left\{ \frac{1}{\sum_{j=0}^{d_i} h_{ij}} \sum_{k=0}^{d_i} k h_{ik} \right\} \tag{6}$$

[3] Node 1 itself is considered reachable in 0 hops.

which simplifies to

$$\bar{h} = \frac{1}{N^2} \sum_{i=1}^{N} \left\{ \sum_{k=0}^{d_i} k h_{ik} \right\} \tag{7}$$

if the network is connected, i.e.,

$$\sum_{k=0}^{d_i} h_{ik} = N \tag{8}$$

for $(1 \le i \le N)$.

Since connectivity is not always assured, define

$$\Gamma_{ik} = \frac{1}{N} \sum_{j=0}^{k} h_{ij} \tag{9}$$

as a measure of nodal connectivity. If a given node is connected then all other nodes are reachable from it and $\Gamma_{i\infty} = 1$. If, however, the node is not connected then $\Gamma_{i\infty} < 1$. This measure is readily extended to include the entire network as follows:

$$\Gamma_k = \frac{1}{N} \sum_{i=1}^{N} \Gamma_{ik}. \tag{10}$$

For a connected network, $\Gamma_\infty = 1$. Otherwise, $\Gamma_{i\infty} < 1$. Γ_k thus provides a measure of network connectivity which can be used in addition to the mean internodal distance.

B. Rings, Manhattan Streets, Hypercubes, Shufflenets and $\bar{h}$

Using the definition of (1) the mean internodal distance $\bar{h}$ for several networks which are often cited in the literature may be derived. In most cases simple analytical results can be obtained.

• *Rings*

A ring network is shown in Fig. 2. It is connected with $p_i^{\text{out}} = p_i^{\text{in}} = 1$ and $d_i = N$ for all i. Thus, $n_{ik} = 1$ for all $1 \le i \le N$ and $1 \le k \le N$ and $\bar{h}$ is readily calculated as $\bar{h} = (N-1)/2$. The ring is minimal in that it uses the fewest links possible to achieve connectivity.

• *Manhattan Street Networks*

A Manhattan Street Network [6] is shown in Fig. 3. N nodes are arranged in a rectangular grid of dimensions $X \times Y = N$. Nodes on the edges are connected to nodes on the opposite edge so that the structure could be mapped onto a torus. Link directions alternate by rows and columns much as do the streets of Manhattan. Thus, X and Y should be even,[4] $p_i^{\text{out}} = p_i^{\text{in}} = p = 2$ and the number of links $L = 2N$. In general, there is no analytic expression for the average distance between nodes in the Manhattan Street Network. However, for the special case of both X and Y divisible by 4, the mean internodal distance is found to be[5]

$$\bar{h} = \frac{X+Y}{4} + 1 - \frac{4}{N}. \tag{11}$$

[4] Except where either X or Y is 1. In this case, the variable not equal to unity is unconstrained.

[5] This result has been verified for $N < 50,000$ by computer calculation but remains unproven analytically.

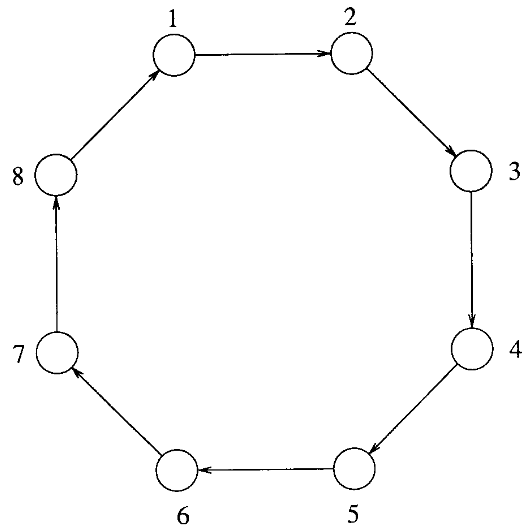


Fig. 2. A ring network with 8 nodes.

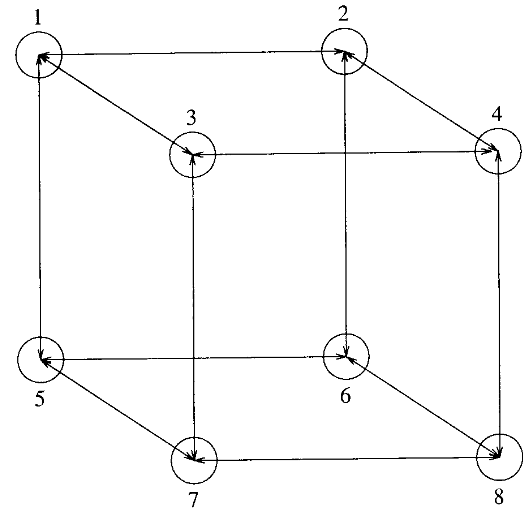


Fig. 4. A hypercube network with 8 nodes and 12 bidirectional links.

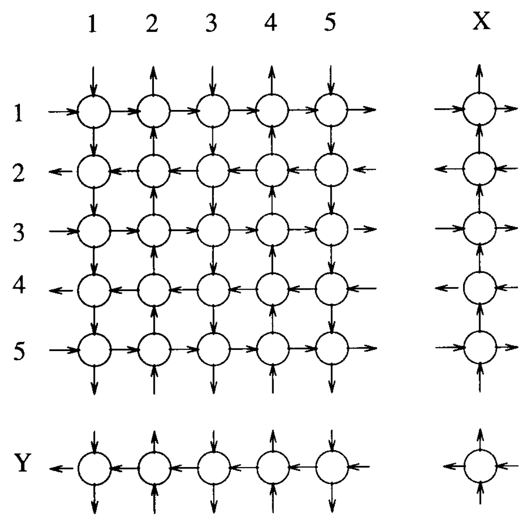


Fig. 3. A Manhattan Street Network with $X \times Y$ nodes. Notice that the directions of the directed links alternate from row to row and from column to column. Also notice that links from nodes on the edges "wrap around" to nodes on the opposite edges.

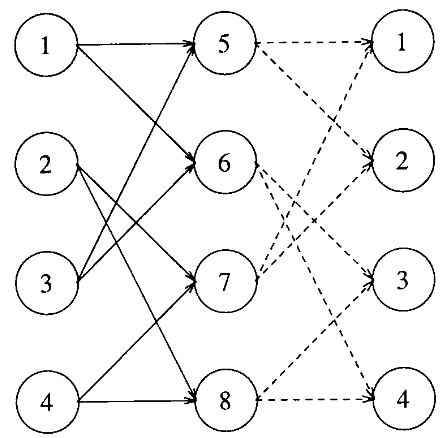


Fig. 5. A Shufflenet with 8 nodes and 16 links. Notice the staged structure and the identical link pattern between stages. Also notice that the structure "wraps around" at the right-most stage.

Thus, for $X = Y$ the mean internodal distance grows as $\sqrt{N}/2$.

• *Hypercubes*

The nodes of a hypercube [8] network are the vertices of k-dimensional hypercube. Thus, the number of nodes in a hypercube network is $N = 2^k$ and the out-degree/in-degree of each node is k. Note that the number of directed links used in $L = Nk$. The mean internodal distance is $h = 1/2k$. A hypercube of degree 3 and eight nodes is shown in Fig. 4.

• *Shufflenets*

The nodes of a Shufflenet are arranged in k columns of p^k nodes. Thus, $N = kp^k$ and the in-degree = out-degree = p. The number of links used is $L = Np$ and the mean internodal distance is given by

$$\bar{h}_{\text{shufflenet}} = \frac{\sum_{i=0}^{k-1} ip^i + \sum_{i=0}^{k} (k+i)(p^k - p^i)}{kp^k} \tag{12}$$

which reduces to

$$\bar{h}_{\text{shufflenet}} = \frac{3k-1}{2} - \frac{p^k - 1}{p^k(p-1)} \tag{13}$$

as similarly derived in [4].[6] A Shufflenet with $p = 2$, $k = 2 (N = 8)$ is shown in Fig. 5.

In Fig. 6, $\bar{h}$ as a function of network size N is plotted for the various networks. The comparison is slightly misleading in that networks with differing number of links per node are compared. However, Fig. 6 does illustrate that networks

[6] In [4] the total network traffic is assumed to be $kp^k - 1$ since traffic from a node to itself is not considered. Such self-traffic is considered here although it is assumed that it requires zero hops to reach its destination.

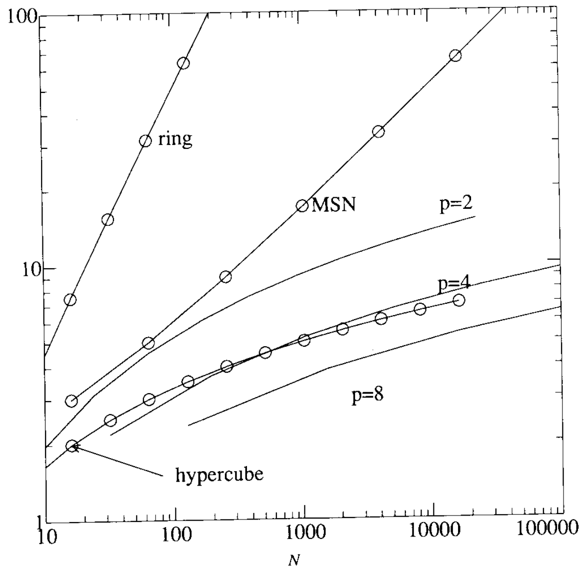


Fig. 6. A comparison of the mean internodal distances for various networks. The ring has the largest $\bar{h}$ followed by the manhattan street network (MSN). The hypercube has seemingly small $\bar{h}$ as compared to the previous two and in some cases as compared to shufflenet (solid lines with no symbols). However, the out-degree of the hypercube is $2\bar{h}$ so that when compared to a shufflenet with the same number of nodes AND comparable out-degree, the hypercube has a higher mean internodal distance. For example, the hypercube curve at (32, 2.5) is a network with 32 nodes and out-degree 5. The shufflenet with out-degree 4 and 32 nodes has substantially lower $\bar{h}$. This effect is amplified with increasing N.

such as the ring and the Manhattan Street Network impose much larger mean internodal distances as N increases. The hypercube imposes similar properties when compared to a Shufflenet (with a comparable number of links per node). Thus Shufflenet appears to be efficient in terms of providing a low-mean internodal distance with few links per node. Therefore, attention will be restricted to Shufflenet for the purposes of comparison.

III. Random and Semi-Random Networks

A. A Description

Consider a set of N nodes and L links. The total number of networks T which may be constructed is[7]

$$T_{\text{random}} = \begin{bmatrix} N^2 \\ L \end{bmatrix}. \tag{14}$$

Any network chosen at random from this set is considered completely random. Notice that even for small N and L, T_{random} is very large.[8] Also notice that over the ensemble of networks, each node has an average of $p = L/N$ incoming links.

However, this definition of completely random networks, in which the number of links is fixed but distributed randomly over the entries in the connection matrix, is difficult to analyze. In also has the disturbing property that a given node may receive or extend absolutely no links. Therefore, a subset of networks was chosen which preserves the general character of randomness while being more analytically tractable and precluding the possibility of complete nodal isolation. Specifically, each node must have exactly p outgoing (or incoming) links [1]. Such networks will be called *semirandom*. The number of possible networks in this case is,

$$T_{\text{semi}} = \begin{bmatrix} N \\ p \end{bmatrix}^N = \begin{bmatrix} N \\ L/N \end{bmatrix}^N. \tag{15}$$

For $N = 8$ and $p = 2$, $T_{\text{semi}} \approx 4 \times 10^{11}$, which is still a very large number.

Notice that no stipulation was made as to the level of connectivity in such networks. In short, both random and semirandom networks may not be connected, although they do in general show high levels of connectivity. Nonetheless, since connectivity is a desirable property of a network, steps can be taken to ensure connectivity. Such networks as are defined here as "connected semirandom networks." Connectivity is ensured by adding links at random to a ring network. Thus, $p - 1$ outgoing (or incoming) links are added at random to each node in the structure.[9] The ring was chosen as the starting point since it uses the fewest links to ensure connectivity. This leaves the maximum number of remaining links thereby maximizing the number of different possible networks. Notice that any network which contains a ring network is Hamiltonian[10] by definintion. This general structure therefore covers a large range of networks of interest. For example, Shufflenet, the Manhattan Street Network (with an even number of columns or rows) and the hypercube are all Hamiltonian. The number of such networks is,

$$T_{\text{connected-semi}} = \begin{bmatrix} N-1 \\ p-1 \end{bmatrix}^N. \tag{16}$$

For $N = 8$ and $p = 2$, $T_{\text{connected-semi}} \approx 6 \times 10^6$ which is substantially smaller than semirandom and random networks, but still reasonably large.

B. Analytic/Experimental Methodology

The properties of semirandom and connected semirandom networks will be examined by considering the connectivity of a single node. Specifically, for a given node, the mean internodal distance and the total number of reachable nodes are computed. If this process is repeated for many different networks, a statistical picture of nodal connectivity of the ensemble of possible networks can be obtained. This idea forms the basis of both the experimental and analytic approach of this paper; analyze the connectivity for many networks from the vantage point of a single node.

C. Analysis of Semirandom Networks

To be rigorous one could start at a given node and produce the probability distribution on the number of new nodes

[7]Corresponding to L nonzero entries in the $N \times N$ connection matrix.

[8]For $N = 8$ and $L = 16$, $T_{\text{random}} \approx 5 \times 10^{14}$.

[9]This structure is similar to the "chordal ring" [10], [9].

[10]A Hamiltonian network contains at least one closed path which covers every node exactly once [1]–[3].

reached in one hop. For a semirandom network the number of new nodes reached in one hop will be either p or $p-1$. Using this one-hop distribution, the distribution of new nodes reached in 2 hops could be obtained. Repeating this process, a probability distribution may be constructed for the number of new nodes reached in any given number of hops. From these hop distributions, the mean internodal distance for the network as well as the variation of this quantity from node to node can be calculated. The probability distribution on the level of connectivity would also be derived in this process.

However, this method is cumbersome for large N owing to the large number of probability distributions which must be calculated. Therefore, an approximation is used (see Appendix). The set of recursive equations describing the mean number of 1) new nodes h_k reached on hop k and 2) Γ_k the fraction of nodes reached up to and including hop k is given below.

$$h_{k+1} = N(1-\Gamma_k)\left\{1-\left(1-\frac{p}{N}\right)^{h_k}\right\} \tag{17}$$

$$\Gamma_{k+1} = \Gamma_k + \frac{h_{k+1}}{N}. \tag{18}$$

These equations may be combined to obtain the composite,

$$\Gamma_{k+1} = 1 + (\Gamma_k - 1)\alpha^{N(\Gamma_k - \Gamma_{k-1})} \tag{19}$$

where $\alpha = (1 - p/N)$.

Fig. 7 shows Γ_k versus k using both the analytic approximation and the average of 100 computer simulations. The agreement is close for various N and p. As could be expected, however, the approximation is less accurate for $N < 100$. The mean internodal distance as a function of network size N for various numbers of links, $L = Np$ is plotted in Fig. 8.[11] Γ_∞ for the semirandom networks is also provided. The characteristics of the Shufflenet are shown for comparison.

For small p and N this comparison is misleading since the semirandom networks are not highly connected on average, i.e., ($p = 2 \rightarrow \Gamma_\infty = 0.8$). However, for larger p and N where $\Gamma_\infty \rightarrow 1.0$, the comparison is reasonable. In these cases it can be seen that the semirandom networks provide mean internodal distances *smaller* than the corresponding Shufflenet. The desire to extend this intriguing result to connected networks with smaller p and N prompted a study of *connected* semirandom networks.

D. Experiments with Connected Semirandom Networks

As mentioned previously, a connected semirandom network may be constructed from a ring by assigning the remaining $p-1$ links per node at random. Unfortunately, the connectivity properties of such networks are difficult to analyze. Thus, a number of semirandom networks were constructed and their average properties explored.

Shown in Fig. 9 are plots of $\bar{h}$ versus N for connected semirandom networks. Shufflenet is also shown for comparison. In keeping with the properties of *unconnected* semirandom networks, the mean internodal distance is comparable to but larger than that of Shufflenet for smaller N. However, the difference in mean internodal distance between Shufflenet and connected semirandom networks decreases monotonically with increasing N so that for $N > 1000$ the semirandom networks have substantially smaller mean internodal distances.

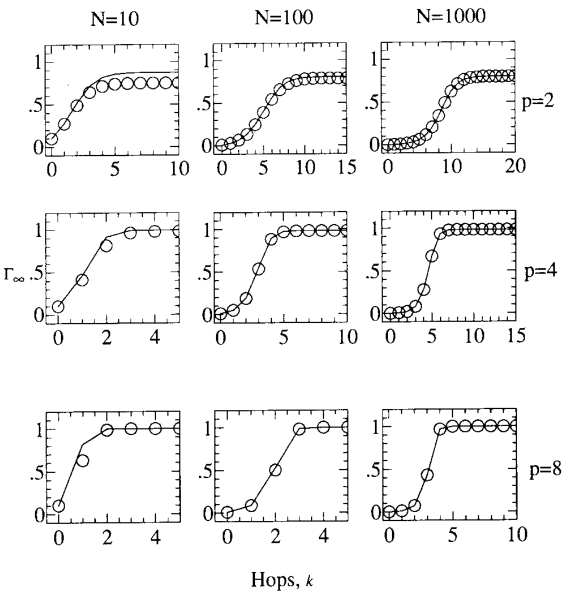


Fig. 7. A montage comparing theoretical and experimental Γ_k curves for unconnected semirandom networks as functions of the number of nodes N and the out-degree p. The theoretical curves are the solid lines while the experimental curves are formed by open circles. Notice the excellent agreement for $N \geq 100$.

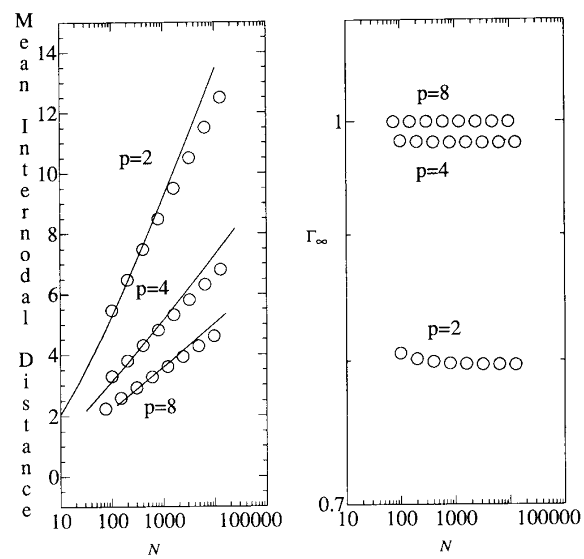


Fig. 8. A comparison of unconnected semirandom networks (open circles) and shufflenets (solid lines) as functions of number of nodes N and outgoing links per node L. For $p = 2$ the comparison is flawed since the semirandom networks are on average unconnected; Γ_∞ is substantially less than 1.0. For $p \geq 4$, however, the comparison is more reasonable since $\Gamma \approx 1.0$. In this case it can be seen that the semirandom nets have comparable $\bar{h}$ for $N < 1000$ and increasingly smaller $\bar{h}$ for $N > 1000$.

[11] The mean internodal distance for the network is the mean of the h_k distribution (suitably normalized of course).

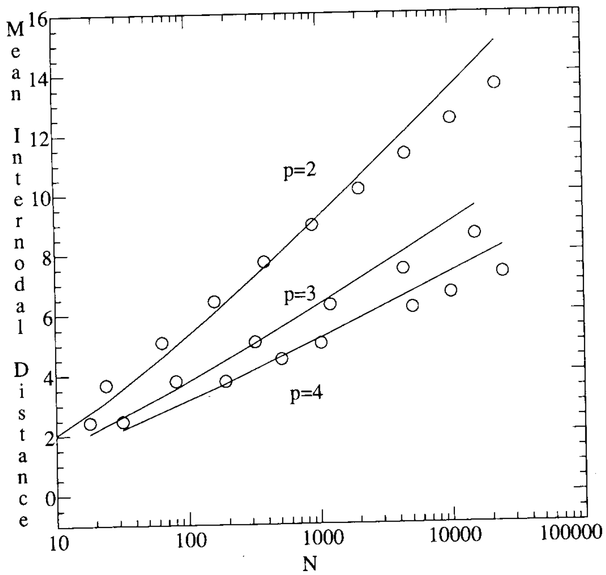


Fig. 9. A comparison of connected semirandom networks (open circles; standard deviation about mean is smaller than symbol size and therefore not shown). with shufflenet (solid lines). The same characteristics seen with the unconnected random networks are evident; for $N > 1000$ the semirandom networks have increasingly smaller mean internodal distances.

Thus, with no network plan other than the initial very general ring structure, a network emerges spontaneously which has mean internodal distance comparable to or better than that of an efficient network such as Shufflenet.

IV. Discussion

A. The Moore Bound

In comparing semirandom networks to Shufflenet an important question has been ignored. Given L links and N nodes, what is the minimum $\bar{h}$ achievable? Although this difficult combinatorial optimization problem has eluded general solution owning to the size and complexity of the space describing such networks, a simple bound may be provided.

Consider a set of N nodes each with out-degree p arranged in a p-ary tree structure. If the level in the tree is denoted by k then $n_k = p^k$ nodes are reached in each level until the final level, f where $N - (p^f - 1)/(p - 1)$ nodes are reached. This tree structure affords the maximum growth in that any only new nodes are reached at every level.[12] Thus, the growth of h_k for any network is strictly bounded by that of the p-ary tree. This result is called the Moore bound [5], [7].[13]

[12] However, there exists no network for which every node is the root of a p-ary tree (a hypothetical but unrealizable Directed Moore Graph [7]).

[13] An important stipulation, implicit in the definition of the Moore bound, is that either the in or out-degree of the nodes be fixed. This condition also guarantees that the complexity of the switching elements within the nodes remains manageable. For example, in a star network one node serves as a central distribution point by taking $N - 1$ inputs from the other $N - 1$ nodes and distributing the appropriate traffic to these same $N - 1$ nodes. The mean internodal distance for this network is less than 2, but the central node is a high-complexity $N - 1 \times N - 1$ switch. In contrast, the switching complexity of the networks considered here is much smaller (please see [13] for further details).

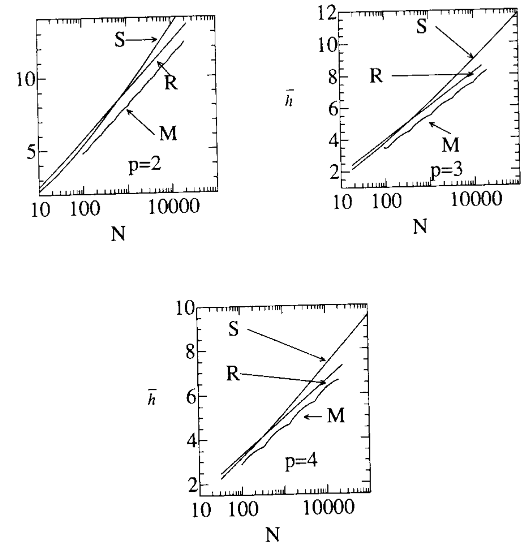


Fig. 10. The mean internodal distances for shufflenet and for connected semirandom networks are compared to the Moore bound, the absolute minimum $\bar{h}$ achievable for and network with p links per node (for p = 2, 3, 4). Notice that the Shufflenet curves diverge from the Moore bound for large N whereas the semirandom curves parallel the Moore bound.

The $\bar{h}$ of the p-ary tree is

$$\bar{h} = \frac{1}{N}\left[\sum_{k=0}^{f-1} kp^k + f\left(N - \frac{p^f - 1}{p - 1}\right)\right]. \tag{20}$$

This bound provides a minimum $\bar{h}$ for a network with p outgoing links per node. In Fig. 10 the Moore bound is compared to Shufflenet and connected semirandom networks and seen to be ≈1 hop lower. Notice the manner in which the Shufflenet curve diverges from the Moore bound for large N whereas the connected semirandom curve parallels the Moore bound.

This raises the question of how closely the Moore bound may be approached. Networks such as the perfect shuffle-exchange [17] are known to provide very low-mean internodal distance. Unfortunately, the variation of mean distance across nodes in the network is large. Thus, the perfect shuffle is "unfair". Nonetheless, the low $\bar{h}$ produced by this network and the almost geometric growth of its h_k is of interest and is considered elsewhere [13].

B. Comments on the Structure of Low $\bar{h}$ Networks

The surprising ability of semirandom networks to provide low-mean internodal distances prompts the question "Why?" One explanation is suggested by examining the h_k versus k dependence of semirandom networks. The h_k versus k for a Shufflenet and a semirandom network of equal size are superimposed in Fig. 11(a). Notice the similarities and differences in the shape of the two curves. Initially, both grow geometrically. However, the Shufflenet curve then abruptly flattens whereas the semirandom network curve continues to grow. Since the "area"[14] under both curves must be identically N, this abrupt flattening leads to a larger mean of the distribution $(\bar{h})$ and greater variation about $\bar{h}$. Conversely, the h_k versus k of the semirandom network does not flatten abruptly yielding a more narrow distribution with a smaller mean value.

This general observation is illustrated in Fig. 11(b) wherein the widths of the h_k distributions for Shufflenet and the connected semirandom networks are plotted as a functions of N for various p. For $N > 1000$ the semirandom networks show significantly smaller variation about an already smaller mean internodal distance.

The flattening of the h_k distribution for Shufflenet is directly attributable to the "staged" structure of Shufflenet in which $N = sp^s$ nodes are arranged in s stages of p^s nodes each. Thus, the maximum number of new nodes reachable on a given hop is p^s and this limit is reached rapidly due to the geometric growth. In contrast, the number of new nodes reachable on a given hop in semirandom networks is not bounded by such a topological constraint as evidenced by the much larger peak of the h_k versus k distribution. Thus, the network designer who desires small $\bar{h}$ should avoid structures with many identical stages if it forces a premature flattening of the h_k curve.

C. Throughput Performance

For regularly constructed networks maximum throughput is usually found by calculating the maximum link load, $l_{\max}$ in response to a uniform traffic distribution. For example, the source associated with each node is assumed to emit one unit of traffic per unit time. This traffic is uniformly distributed to the rest of the network. The mean amount of time messages must stay in the network to reach their destination (sojourn time) can then to used to determine the necessary aggregate link capacity of the network. If each link traversal is assumed to require one unit of time, then $\bar{h}$ is the mean sojourn time. Thus, the required aggregate link capacity link capacity is given by

$$C_l = N\bar{h}. \tag{21}$$

The average link capacity is then

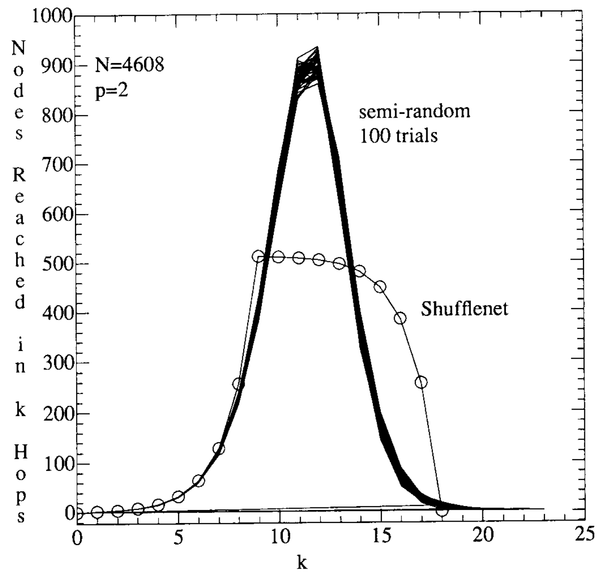
$$\bar{l} = N\bar{h}/L. \tag{22}$$

The normalized throughput is then given by

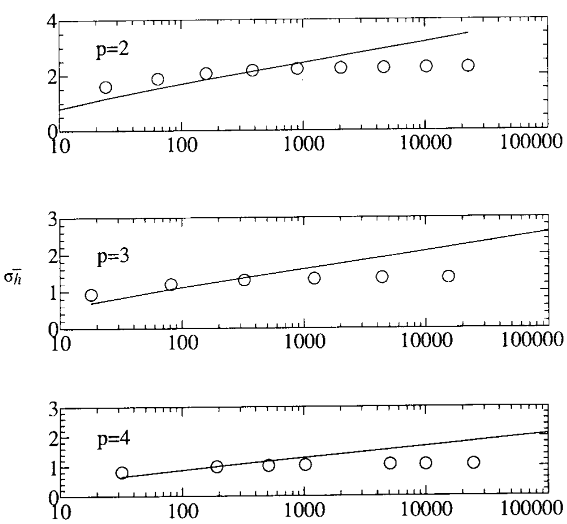
$$T = \bar{l}/l_{\max}. \tag{23}$$

The tacit assumption in this calculation is that all network links should be of equal capacity. Thus, the deviation of any one link load from the nominal mean link load $\bar{l}$ leads to decreased throughput. Regular networks such as Shufflenet can achieve perfect throughputs of $T = 1.0$. In contrast, a semirandom network does not generally achieve such perfect throughput. In limited tests, semirandom networks had poor $(T \approx 0.2)$ throughput performance. This poor performance may be qualitatively attributed to many individual nodes of a semirandom network having unequal numbers of incoming

[14]By "area" it is meant the summation $\sum_{k=0}^{\infty} h_k$.



(a)



(b)

Fig. 11(a). A comparison of h_k for shufflenet (solid line with circles) and 100 simulations of connected semiradom networks (solid lines). $N = 4608$, $p = 2$. Notice that the simulation curves do not differ appreciably from each other. Also notice that the ensemble of h_k for the semirandom networks is "peakier" and narrower than the shufflenet curve. This property tends to produce smaller mean internodal distances and tighter distributions about that mean. Also notice that the ensemble for the 100 simulation trials is tightly packed. This property suggests that there is little variation from node to node in the mean distance to the rest of the network. The semirandom networks are thus reasonably "fair" in providing access to the network. (b) The width $(\sigma_{\bar{h}})$ of the h_k curves for shufflenet (solid lines) and connected semirandom networks (open circles) are plotted as a function of N for the values of $p = 2, 3,$ and 4. The width of the h_k curve provides a measure of the variability the mean internodal distance from a given node. For $N < 1000$ the widths are comparable but as $N \rightarrow \infty$ the width for shufflenet increases whereas that for the semirandom networks seems to stay constant.

and outgoing links. This unbalanced link input and output might lead to variations in link loading and thus to decreased throughput.

It should also be noted that the link loading variation depends upon the routing scheme used. Although a number of different schemes were employed in the tests, no search for the optimal routing pattern was performed. Nonetheless, the performance of the different routing schemes did **not** show tremendous variation. Furthermore, the same routing schemes employed on Shufflenets achieved throughputs of $T \approx 0.9$: leading to the tentative conclusion that the structure of the semirandom network, not the routing scheme, was the deciding factor.

However, the notion of throughput based upon equal capacity links may be more an analytic expedient for regular network rather than an engineering reality; especially in the case of modern fiber optic systems. Older technologies such as copper cable and microwave links, once installed, were of virtually fixed capacity. Adding substantial capacity might entail laying another cable or redesigning a microwave antenna: both nontrivial physical plant investments. Since networks must usually grow with time, this difficulty of increasing capacity may have provided an impetus for maximizing the capacity of each and every link rather than sizing to suit current needs.

This difficulty may not be relevant for fiber optic systems since the fiber itself is of virtually infinite capacity. Any changes in capacity may be accomplished by changing the receiver/transmitter pair attached to the fiber ends or by simply adding pairs at different frequencies as in a wavelength division multiplexing scheme. Thus, providing links of differing sizes may not be at all problematic.

Since the necessary aggregate capacity of a network is directly proportional to the mean amount of time (the mean internodal distance assuming one time unit per hop) messages stay in the network, networks with the smallest mean internodal distance require the smallest aggregate capacity. Since greater bandwidth implies greater cost, the total bandwidth used by the network (the aggregate capacity) should be kept to a minimum. Alternately, for a given aggregate capacity, more traffic may be carried if the network has a smaller mean internodal distance. When need-based link sizing can be reasonably employed, these criteria favor the semirandom network in many cases.

D. Routing Complexity

Another possibly detrimental property of semirandom networks is their irregularity. More regular networks such as Shufflenet can use simple routing algorithms [11], [12]. For an irregular network an explicit routing table is required at each node rather than a simple algorithm. The need for a routing table at each node, however, may not be too onerous since at worst, the memory burden at each node is proportional to N. In addition, if stress is applied to a network wherein links are caused to fail, then having some knowledge of the network topology at each node is a valuable asset regardless of the network structure.

Other criteria important for routing such as the availability of alternate paths in the event of congestion [14], [15], [16] or link failure have not yet been considered. However, in light of the result that the mean internodal distance under deflection routing seems to be proportional to the normal mean internodal distance [14], semirandom networks should perform well using deflection routing.

V. Conclusion

It has been shown that most multihop networks with N nodes, L links and fixed out-degree display surprisingly low-mean internodal distances. It is therefore a relatively simple task to find networks with mean internodal distances comparable to or surpassing those of more regular networks. For this reason, careful selection of network topology to minimize the mean internodal distance as well as the necessary aggregate link capacity may be important in only the most sensitive applications. And even in such sensitive applications, an almost randomly chosen network may be the optimal choice.

In addition, semirandom networks may be particularly well-matched to fiber optic networks. The capacity of a fiber optic cable is primarily constrained by the speed and number (assuming wavelength-division multiplexing) of transmitter/receiver pairs connected to its ends. In microwave or copper cable systems the transmission medium itself is a limiting factor. Thus, uneven distribution of capacity by proper distribution of transmission/reception resources may be more readily achievable in fiber optic systems. Since the link loading in a semirandom network can be reasonably uneven, fiber optics and semirandom networks could prove a good match; procuring networks with minimum aggregate capacity (number of transmitter/receivers) is a reasonable design criterion. Alternatively, the low-mean internodal distance provided by semirandom networks would allow more traffic to be carried for a given aggregate capacity.

Appendix: An Approximation to h_k for Semirandom Networks

Consider the $N \times N$ connection matrix associated with an N node network. Assume that each entry has a probability of p/N of being nonzero. Then each row and column of the matrix has an average of p nonzero entries. This matrix approximates the semirandom network connection matrix with N nodes and $p = L/N$ outgoing links per row as well as the completely random network with exactly L links distributed over the N nodes.

Now consider an arbitrary row of the matrix corresponding to some network node. Given that this node is already covered, the average number of new nonzero entries in that row is $(N-1)p/N$. Thus, the average number of new nodes reached in a single hop is

$$h_1 = (N-1)p/N. \tag{A.1}$$

The average percentage of nodes covered in one hop is then

$$\Gamma_1 = 1/N + (N-1)p/N^2 = \Gamma_0 + h_1/N. \tag{A.2}$$

To calculate the average number of new nodes reached in 2 hops consider the h_1 nodes reached in the previous hop. Notice that if any of these h_1 nodes has a nonzero entry in a previously uncovered column of the connection matrix, then that column is added to the total of new nodes reached in 2 hops. The probability that a given column is covered by nonzero entries in the rows of the h_1 nodes is $1 - \text{Prob(notcovered)} = 1 - (1 - pN)^{h_1}$. Thus, the average number of nodes reached in 2 hops is

$$h_2 = \left(N(1-\Gamma_1)(1-(1-p/N)^{h_1}\right). \tag{A.3}$$

The basic idea is generalized in the following set of equations:

$$h_{k+1} = N(1-\Gamma_k)\left[1 - \left(1 - \frac{p}{N}\right)^{h_k}\right] \tag{A.4}$$

$$\Gamma_{k+1} = \Gamma_k + \frac{h_{k+1}}{N}. \tag{A.5}$$

Combining equations (A.4) and (A.5), recursive equations in either Γ or h may be obtained.

$$h_{k+1} = \left(N - \sum_{i=0}^{k} h_i\right)\left[1 - \left(1 - \frac{p}{N}\right)^{h_k}\right] \tag{A.6}$$

$$\Gamma_{k+1} = 1 + (\Gamma_k - 1)\left(1 - \frac{p}{N}\right)^{N(\Gamma_k - \Gamma_{k-1})} \tag{A.7}$$

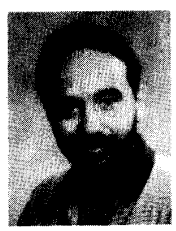
Equation (A.6) is subject to the initial condition of $h_0 = 1$ and (A.7) is subject to $\Gamma_{-1} = 0$ and $\Gamma_0 = 1/N$.

ACKNOWLEDGMENT

I would like to thank the members of the Network Systems Research Department at AT&T Bell Laboratories for their helpful comments and suggestions at the inception of this work. In particular: E. Ayanoglu, I. Bar-David, K. Eng, C. L. I, M. Karol, and C. Woodworth. I would also like to thank two anonymous reviewers for their helpful comments.

REFERENCES

[1] B. Bollobas, *Random Graphs.* New York: Academic, 1985.
[2] E. M. Palmer, *Graphical Evolution.* New York: Wiley-Interscience, 1985.
[3] J. L. Gross and T. W. Tucker, *Topological Graph Theory.* Wiley-Interscience, 1987.
[4] A. S. Acampora, M. J. Karol, and M. G. Hluchyj, "Terabit lightwave networks: The Multihop Approach," *AT&T Tech. J.*, vol. 66, no. 6, pp. 21–34, 1987.
[5] M. G. Hluchyj, and M. J. Karol, "ShuffleNet: An application of generalized perfect shuffles to multihop lightwave networks," *INFOCOM 1988*, New Orleans, LA., Mar. 1988.
[6] N. F. Maxemchuk, "Regular mesh topologies in local and metropolitan area networks," *AT&T Tech. J.*, vol. 64, no. 7, pp. 1659–1685, 1985.
[7] W. G. Bridges and S. Toueg, "On the impossibility of directed moore graphs," *J. Combinatorial Theory (B)*, vol. 29, pp. 339–341, 1980.
[8] H. J. Siegel, "Interconnection networks for SIMD machines," *Comput.* pp. 57–65, June 1979.
[9] K. W. Doty, "New designs for dense processor interconnection networks," *IEEE Trans. Comput.*, vol. C-33, pp. 447–450, 1984.
[10] B. Arden and H. Lee, "Analysis of chordal ring network," *IEEE Trans. Comput.*, vol. C-30, pp. 291–295, Apr. 1981.
[11] A. S. Acampora & M. J. Karol, "An overview of lightwave packet networks," *IEEE Network Mag.*, vol. 3, 1989.
[12] M. J. Karol and S. Shaikh, "A simple adaptive routing scheme for shufflenet multihop lightwave networks," *IEEE Globecom*, Nov. 28, 1988–Dec. 1, 1988.
[13] C. Rose, "Low-mean internodal distance network topologies through simulated annealing," *IEEE Trans. Commun.*, vol. 40, pp. 1319–1326, Aug. 1992.
[14] E. Ayanoglu and R. J. Caballero, "Signal flow graphs for the delay analysis of deflection routing in multihop networks," preprint, 1989.
[15] A. G. Greenberg and B. Hajek, "Deflection routing in hypercube networks," preprint, 1989.
[16] A. Krishna and B. Hajek, "Performance of shuffle-like switching networks with deflection," preprint, 1989.
[17] H. S. Stone, "Parallel processing with the perfect shuffle," *IEEE Trans. Comput.*, vol. C-20, pp. 153–161, 1971.



Christopher Rose (M'92) was born on January 9, 1957 in New York City. He received the B.S., M.S., and Ph.D. degrees from the Massachusetts Institute of Technology in Cambridge, MA, in 1979, 1981, and 1985, respectively. His graduate work was supported by the AT&T Cooperative Research Fellowship Program.

Following graduate school he was with AT&T Bell Laboratories in Holmdel, N.J. as a member of Network Systems Research Department. He is currently an Assistant Professor of Electrical & Computer Engineering at Rutgers University, NJ. His current technical interests include the dynamics of large communications networks, mobile communications networks, cellular automata as well as applications of high-temperature superconductivity to communications problems.

Dr. Rose is a Henry Rutgers Research Fellow at Rutgers University.