A large deformation theory for rate-dependent elastic–plastic materials with combined isotropic and kinematic hardening

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**Abstract**

We have developed a large deformation viscoplasticity theory with combined isotropic and kinematic hardening based on the dual decompositions \( F = F^e F^p \) [Kröner, E., 1960. Allgemeine kontinuumstheorie der versetzungen und eigenspannungen. Archive for Rational Mechanics and Analysis 4, 273–334] and \( F^p = F^p_{en} F^p_{dis} \) [Lion, A., 2000. Constitutive modelling in finite thermoviscoelasticity: a physical approach based on nonlinear rheological models. International Journal of Plasticity 16, 469–494]. The elastic distortion \( F^e \) contributes to a standard elastic free-energy \( w(e) \), while \( F^p_{en} \), the energetic part of \( F^p \), contributes to a defect energy \( w(p) \) – these two additive contributions to the total free energy in turn lead to the standard Cauchy stress and a back-stress. Since \( F = FF^p \) and \( F^p_{en} = FF^p_{dis} \), the evolution of the Cauchy stress and the back-stress in a deformation-driven problem is governed by evolution equations for \( F^p_{en} \) and \( F^p_{dis} \) – the two flow rules of the theory.

We have also developed a simple, stable, semi-implicit time-integration procedure for the constitutive theory for implementation in displacement-based finite element programs. The procedure that we develop is “simple” in the sense that it only involves the solution of one non-linear equation, rather than a system of non-linear equations. We show that our time-integration procedure is stable for relatively large time steps, is first-order accurate, and is objective.

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**1. Introduction**

In classical small deformation theories of metal plasticity, when attempting to model the Bauschinger phenomenon (Bauschinger, 1886) and other phenomena associated with cyclic loading, kinet-
matic strain hardening is often invoked by introducing a symmetric and deviatoric stress-like tensorial internal variable $\mathbf{T}_{\text{back}}$ called the back-stress, which acts to oppose the Cauchy stress $\mathbf{T}$ in the formulation of an appropriate yield criterion and a corresponding flow rule for the material. The simplest model for the evolution of such a back-stress is Prager’s linear kinematic-hardening rule (Prager, 1949):

$$\mathbf{T}_{\text{back}} = B \mathbf{E}^p \text{ so that } \mathbf{T}_{\text{back}} = B \mathbf{E}^p,$$

where $B > 0$ is a back-stress modulus and $\mathbf{E}^p$ is the plastic strain tensor in the standard decomposition $\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p$ of the infinitesimal strain tensor $\mathbf{E}$ into elastic and plastic parts. A better description of kinematic hardening is given by an evolution equation for the back-stress $\mathbf{T}_{\text{back}}$ proposed initially by Armstrong and Frederick (1966): \(^1\)

$$\mathbf{T}_{\text{back}} = B \mathbf{E}^p - \gamma \mathbf{T}_{\text{back}} \mathbf{\dot{E}}^p, \quad \text{with } \mathbf{\dot{E}}^p \text{ def } |\mathbf{\dot{E}}^p|,$$

where the term $(\gamma \mathbf{T}_{\text{back}} \mathbf{\dot{E}}^p)$ with $\gamma > 0$ represents a dynamic recovery term, which is collinear with the back-stress $\mathbf{T}_{\text{back}}$ and is proportional to the equivalent plastic strain rate $\mathbf{\dot{E}}^p$. For monotonic uniaxial loading, the evolution of back-stress, instead of being linear as prescribed by (1.1), is then exponential with a saturation value. The Armstrong–Frederick model for kinematic hardening is widely used as a simple model to describe cyclic loading phenomena in metals. For a discussion of the Armstrong–Frederick model and its various extensions to model cyclic plasticity in the small deformation regime see Lemaitre and Chaboche (1990) and the recent review by Chaboche (2008).

While there have been numerous attempts to extend the Armstrong–Frederick rule to large deformations, a suitable extension is still not widely agreed upon. Dettmer and Reese (2004) have recently reviewed and numerically analyzed several previous models for large deformation kinematic hardening of the Armstrong–Frederick type. One class of theories (cf., Dettmer and Reese, 2004, for a list of references to the literature) follow Prager and Armstrong and Frederick, and introduce a stress-like internal variable to model kinematic hardening; these theories require suitable frame-indifferent evolution equations to generalize an evolution equation resembling (1.2) to large deformations. An alternative to such theories are theories based on a proposal by Lion (2000), who in order to account for energy storage mechanisms associated with plastic flow, introduced a kinematic constitutive assumption that the plastic distortion $\mathbf{F}^p$ in the standard Kröner (1960) elastic–plastic decomposition $\mathbf{F} = \mathbf{F}^p + \mathbf{F}^e$ of the total deformation gradient, may be further multiplicatively decomposed into energetic and dissipative parts $\mathbf{F}_{\text{en}}^p$ and $\mathbf{F}_{\text{dis}}^p$, respectively, as $\mathbf{F}^p = \mathbf{F}_{\text{en}}^p \mathbf{F}_{\text{dis}}^p$ (cf., Lion’s Fig. 5). \(^2\) The elastic distortion $\mathbf{F}^e$ contributes to a standard elastic free-energy $\psi^{(0)}$, while $\mathbf{F}_{\text{en}}^p$ contributes a defect energy $\psi^{(1)}$ – these two additive contributions to the total free energy in turn lead to the standard Cauchy stress and a back-stress. Since $\mathbf{F}^p = \mathbf{F}^{p-1} \mathbf{F}^e$ and $\mathbf{F}_{\text{en}}^p = \mathbf{F}_{\text{en}}^{p-1} \mathbf{F}^e$, the evolution of the Cauchy stress and the back-stress in a deformation-driven problem is governed by evolution equations for $\mathbf{F}^e$ and $\mathbf{F}_{\text{dis}}^p$ – the two flow rules of the theory. An important characteristic of such a theory is that since $\mathbf{F}^e$ is invariant under a change in frame, then so also are $\mathbf{F}_{\text{en}}^p$ and $\mathbf{F}_{\text{dis}}^p$ and the resulting evolution equations for the back-stress are automatically frame-indifferent, and do not require special considerations such as those required when generalizing equations of the type (1.2) to proper frame-indifferent counterparts. Examples of kinematic-hardening constitutive theories based on Lion-type $\mathbf{F}^p = \mathbf{F}_{\text{en}}^p \mathbf{F}_{\text{dis}}^p$ decomposition may be found in Dettmer and Reese (2004), Menzel et al. (2005), Håkansson et al. (2005), Wallin and Ristinmaa (2005), and Vladimirov et al. (2008).

While Dettmer and Reese (2004) in Section 4.3 of their paper, and more recently Vladimirov et al. (2008) in Section 2.2 of their paper have presented a generalization of the Armstrong–Frederick type of kinematic-hardening rate-independent plasticity theory using the Lion decomposition $\mathbf{F}^p = \mathbf{F}_{\text{en}}^p \mathbf{F}_{\text{dis}}^p$, we find their presentation rather brief, and (in our view) not sufficiently expository. The primary focus of the papers by Dettmer and Reese and Vladimirov et al., was the development of suitable time-integration algorithms, and they have developed several fully-implicit time-integration procedures for

\(^1\) Also recently published as Frederick and Armstrong (2007).

\(^2\) Lion uses the notation $\mathbf{F}_{\text{en}} = \mathbf{F}, \mathbf{F}_{\text{dis}} = \mathbf{F}^p, \mathbf{F}_{\text{en}} = \mathbf{F}^p, \mathbf{F}_{\text{en}} = \mathbf{F}^e, \mathbf{F}_{\text{en}} = \mathbf{F}^{p-1} \mathbf{F}^e$. Based on a statistical-volume-averaging argument for the response of a polycrystalline aggregate, Clayton and McDowell (2003, Eq. (13)) have also introduced a Lion-type decomposition $\mathbf{F}^p = \mathbf{F}_{\text{en}}^p \mathbf{F}_{\text{en}}^p$; they call $\mathbf{F}^e$ a "meso-incompatibility tensor."
their model; one such integration procedure involves the solution of a system of 13 highly non-linear equations – cf., Eq. (53) of Vladimirov et al. (2008). Motivated by the work of these authors, it is the purpose of this paper to:

1. Develop a thermodynamically consistent rate-dependent plasticity theory with combined isotropic and kinematic hardening of the Armstrong–Frederick type, based on the dual decompositions \( \mathbf{F} = \mathbf{F}^e \mathbf{F}^p \) and \( \mathbf{F}^e = \mathbf{F}^e_{\text{en}} \mathbf{F}^e_{\text{sym}} \). The limit \( m \to 0 \), where \( m \) is a material rate-sensitivity parameter in our theory, corresponds to a rate-independent model. Importantly, we develop the theory based on the principle of virtual power and carefully lay down all assumptions and specializations that we have adopted so that it may in the future be possible to generalize the theory with other, more flexible isotropic and kinematic-hardening models.

2. Develop a simple, stable, semi-implicit time-integration procedure for our rate-dependent constitutive theory for implementation in displacement-based finite element programs. The procedure that we develop is “simple” in the sense that it involves the solution of only one stiff non-linear equation, rather than a system of non-linear equations. We show that our time-integration procedure is stable for relatively large time steps, is first-order accurate, and is objective.

2. Kinematics

We consider a homogeneous body \( B \) identified with the region of space it occupies in a fixed reference configuration, and denote by \( X \) an arbitrary material point of \( B \). A motion of \( B \) is then a smooth one-to-one mapping \( x = \chi(X,t) \) with deformation gradient, velocity and velocity gradient given by

\[
\mathbf{F} = \nabla \chi, \quad \mathbf{v} = \dot{\chi}, \quad \mathbf{L} = \text{grad} \mathbf{v} = \mathbf{F}^{-1}. \tag{2.1}
\]

To model the inelastic response of materials, we assume that the deformation gradient \( \mathbf{F} \) may be multiplicatively decomposed as (Kröner, 1960)

\[
\mathbf{F} = \mathbf{F}^e \mathbf{F}^p. \tag{2.2}
\]

As is standard, we assume that \( J = \det \mathbf{F} > 0 \), and consistent with this we assume that

\[
J^e = \det \mathbf{F}^e > 0, \quad J^p = \det \mathbf{F}^p > 0, \tag{2.3}
\]

so that \( \mathbf{F}^e \) and \( \mathbf{F}^p \) are invertible. Here, suppressing the argument \( t \):

- \( \mathbf{F}^e(X) \) represents the local inelastic distortion of the material at \( X \) due to a “plastic mechanism.” This local deformation carries the material into – and ultimately “pins” the material to – a coherent structure that resides in the structural space \( ^5 \) at \( X \) (as represented by the range of \( \mathbf{F}^e(X) \));
- \( \mathbf{F}^p(X) \) represents the subsequent stretching and rotation of this coherent structure, and thereby represents the corresponding “elastic distortion.”

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3 With reference to the power-law model (9.35) in Section 9.3, the stiffness of the equations depends on the strain-rate-sensitivity parameter \( m \), and the stiffness increases to infinity as \( m \) tends to zero, the rate-independent limit. For small values of \( m \) special care is required to develop stable constitutive time-integration procedures; cf., e.g., Lush et al. (1989).

4 Notation: We use standard notation of modern continuum mechanics. Specifically: \( \nabla \) and \( \text{Div} \) denote the gradient and divergence with respect to the material point \( X \) in the reference configuration; \( \text{grad} \) and \( \text{div} \) denote these operators with respect to the point \( x = \chi(X,t) \) in the deformed body; a superposed dot denotes the material time-derivative. Throughout, we write \( \mathbf{F}^{-1} = (\mathbf{F}^e)^{-1}, \mathbf{F}^e_{\text{en}} = (\mathbf{F}^e)^{-1}, \mathbf{F}^e_{\text{sym}} = (\mathbf{F}^e)^{-1}, \mathbf{F}^p_{\text{en}} = (\mathbf{F}^p)^{-1}, \mathbf{F}^p_{\text{sym}} = (\mathbf{F}^p)^{-1} \). We write \( \text{trA}, \text{sym}A, \text{skw}A, A_0, \text{sym}_0A, \) respectively, for the trace, symmetric, skew, deviatoric, and symmetric-deviatoric parts of a tensor \( A \). Also, the inner product of tensors \( A \) and \( B \) is denoted by \( A \cdot B \), and the magnitude of \( A \) by \( |A| = \sqrt{A \cdot A} \).

5 Also sometimes referred to as the “intermediate” or “relaxed” local space at \( X \).
We refer to $F^e$ and $F^p$ as the inelastic and elastic distortions.

By (2.1), and (2.2),

$$L = L^e + F^e L^p F^{-1},$$

with

$$L^e = F^e F^{-1}, \quad L^p = F^p F^{-1}.$$  \tag{2.5}

As is standard, we define the total, elastic, and plastic stretching and spin tensors through

$$D = \text{sym} L, \quad W = \text{skw} L,$$

$$D^e = \text{sym} L^e, \quad W^e = \text{skw} L^e,$$

$$D^p = \text{sym} L^p, \quad W^p = \text{skw} L^p,$$ \tag{2.6}

so that $L = D + W$, $L^e = D^e + W^e$, and $L^p = D^p + W^p$.

The right and left polar decompositions of $F$ are given by

$$F = RU = VR,$$  \tag{2.7}

where $R$ is a rotation (proper-orthogonal tensor), while $U$ and $V$ are symmetric, positive-definite tensors with

$$U = \sqrt{F^T F}, \quad V = \sqrt{F F^T}.$$  \tag{2.8}

Also, the right and left Cauchy–Green tensors are given by

$$C = U^2 = F^T F, \quad B = V^2 = FF^T.$$  \tag{2.9}

Similarly, the right and left polar decompositions of $F^e$ and $F^p$ are given by

$$F^e = R^e U^e = V^e R^e, \quad F^p = R^p U^p = V^p R^p,$$ \tag{2.10}

where $R^e$ and $R^p$ are rotations, while $U^e$, $V^e$, $U^p$, $V^p$ are symmetric, positive-definite tensors with

$$U^e = \sqrt{F^e T F^e}, \quad V^e = \sqrt{F^e F^e T}, \quad U^p = \sqrt{F^p T F^p}, \quad V^p = \sqrt{F^p F^p T}.$$ \tag{2.11}

Also, the right and left elastic Cauchy–Green tensors are given by

$$C^e = U^2 = F^e T F^e, \quad B^e = V^2 = F^e F^e T,$$ \tag{2.12}

and the right and left plastic Cauchy–Green tensors are given by

$$C^p = U^2 = F^p T F^p, \quad B^p = V^2 = F^p F^p T.$$ \tag{2.13}

### 2.1. Incompressible, irrotational plastic flow

We make two basic kinematical assumptions concerning plastic flow:

(i) First, we make the standard assumption that plastic flow is incompressible, so that

$$j^p = \det F^p = 1 \quad \text{and} \quad \text{tr} L^p = 0.$$ \tag{2.14}

Hence, using (2.2) and (2.14),

$$j^e = j.$$ \tag{2.15}
(ii) Second, from the outset we constrain the theory by limiting our discussion to circumstances under which the material may be idealized as isotropic. For isotropic elastic-viscoplastic theories utilizing the Kröner decomposition it is widely assumed that the plastic flow is irrotational in the sense that

\[ W^p = 0. \]  

(2.16)

Then, trivially, \( L^p \equiv D^p \) and

\[ F^p = D^p F^p, \quad \text{with} \quad \text{tr} D^p = 0. \]  

(2.17)

Thus, using (2.1), (2.4), (2.5), and (2.17), we may write (2.4) for future use as

\[ \text{grad} \, v = L^e + F^p D^p F^{p^{-1}}, \quad \text{with} \quad \text{tr} D^p = 0. \]  

(2.18)

2.2. Decomposition of \( F^p \) into energetic and dissipative parts

Next following Lion (2000), in order to account for energy storage mechanisms associated with plastic flow, we introduce an additional kinematic constitutive assumption that \( F^p \) may be multiplicatively decomposed into an energetic \( F^p_{\text{en}} \) and a dissipative part \( F^p_{\text{dis}} \) as follows:

\[ F^p = F^p_{\text{en}} F^p_{\text{dis}}, \quad \text{det} F^p_{\text{en}} = \det F^p_{\text{dis}} = 1. \]  

(2.19)

We call the range of \( F^p_{\text{en}}(X) \) the local substructural space. Thus \( F^p_{\text{en}} \) maps material elements from the substructural space to the structural space (which is the range of \( F^p(X) \)). Lion (2000) calls the substructural space as the “intermediate configuration of kinematic hardening.” A “physical interpretation” of such an “intermediate configuration” is not completely clear; here we simply treat it as a mathematical construct resulting from Lion’s presumed decomposition (2.19).

Then, from (2.5)

\[ L^p = L^p_{\text{en}} + F^p_{\text{en}} L^p_{\text{dis}} F^{p^{-1}}_{\text{en}} \]  

(2.20)

with

\[ L^p_{\text{en}} = F^p_{\text{en}} F^{p^{-1}}_{\text{en}} \quad \text{with} \quad \text{tr} L^p_{\text{en}} = 0, \quad L^p_{\text{dis}} = F^p_{\text{dis}} F^{p^{-1}}_{\text{dis}} \quad \text{with} \quad \text{tr} L^p_{\text{dis}} = 0, \]  

(2.21)

and corresponding stretching and spin tensors through

\[ \bar{F}^p = Q F^p, \quad \bar{F}^p = F^p Q^T, \]

in which \( Q(X, t) \) is an arbitrary time-dependent rotation of the structural space. Unfortunately, Green and Naghdi viewed structural frame-indifference as a general principle; that is, a principle that stands at a level equivalent to that of conventional frame-indifference. This view has been refuted by many workers (cf., e.g., Dafalias, 1998). Based on the classical principle of frame-indifference (cf., Section 3), Green and Naghdi (1971) (cf., also Casey and Naghdi, 1980), introduced the notion of a change in frame of the intermediate structural space. This notion leads to transformation laws for \( F^p \) and \( F^p \) of the form

\[ \bar{F}^p = Q F^p, \quad \bar{F}^p = F^p Q^T, \]

\[ \text{in which} \quad Q(X, t) \text{is an arbitrary time-dependent rotation of the structural space. Unfortunately, Green and Naghdi viewed structural frame-indifference as a general principle; that is, a principle that stands at a level equivalent to that of conventional frame-indifference. This view has been refuted by many workers (cf., e.g., Dafalias, 1998, and the references therein). While we agree with the view that structural frame-indifference is not a general principle, this hypothesis may be used to represent an important facet of the behavior of a large class of polycrystalline materials based on the Kröner decomposition. Indeed, for polycrystalline materials without texture the structural space is associated with a collection of randomly oriented lattices, and hence the evolution of dislocations through that space should be independent of the frame with respect to which this evolution is measured. Using the notion of a change in frame of the structural space, Gurtin and Anand (2005, p. 1714) have shown that within a framework for isotropic plasticity based on the Kröner decomposition (which we follow here), one may assume, without loss in generality, that the plastic spin vanishes \( W^p = 0 \).

6 A note on role of the plastic spin in isotropic solids: There are numerous publications discussing the role of plastic spin in theories based on the Kröner decomposition (cf., e.g., Dafalias, 1998). Based on the classical principle of frame-indifference (cf., Section 3), Green and Naghdi (1971) (cf., also Casey and Naghdi, 1980), introduced the notion of a change in frame of the intermediate structural space. This notion leads to transformation laws for \( F^p \) and \( F^p \) of the form

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7 However, see Clayton and McDowell (2003) for a “statistical-volume-averaging” argument for the response of a polycrystalline aggregate.
\[
\begin{align*}
\mathbf{D}_{en}^p &= \text{sym} \mathbf{L}_{en}^p, \quad \mathbf{W}_{en}^p = \text{skw} \mathbf{L}_{en}^p, \\
\mathbf{D}_{dis}^p &= \text{sym} \mathbf{L}_{dis}^p, \quad \mathbf{W}_{dis}^p = \text{skw} \mathbf{L}_{dis}^p,
\end{align*}
\]  
(2.22)

so that \( \mathbf{L}_{en}^p = \mathbf{D}_{en}^p + \mathbf{W}_{en}^p \) and \( \mathbf{L}_{dis}^p = \mathbf{D}_{dis}^p + \mathbf{W}_{dis}^p \).

Paralleling the assumption of plastic irrotationality for isotropic materials, \( \mathbf{W}^p = \mathbf{0} \), we assume that \( \mathbf{W}_{dis}^p = \mathbf{0} \).  
(2.23)

Then, trivially \( \mathbf{L}_{dis}^p = \mathbf{D}_{dis}^p \), and 
\[
\mathbf{F}^p_{dis} = \mathbf{D}^p_{dis} \mathbf{F}^p_{dis} \quad \text{with} \quad \text{tr} \mathbf{D}^p_{dis} = 0.
\]  
(2.24)

Since
\[
\mathbf{W}^p = \mathbf{W}_{en}^p + \text{skw} \left( \mathbf{F}^p_{en} \mathbf{D}^p_{dis} \mathbf{F}^p_{en}^{-1} \right),
\]  
(2.25)
in order to enforce the constitutive constraint \( \mathbf{W}^p = \mathbf{0} \), we require that both 
\[
\mathbf{W}_{en}^p = \mathbf{0},
\]  
(2.26)

and
\[
\text{skw} \left( \mathbf{F}^p_{en} \mathbf{D}^p_{dis} \mathbf{F}^p_{en}^{-1} \right) = \mathbf{0}.
\]  
(2.27)

Also, (2.20) reduces to
\[
\mathbf{D}^p = \mathbf{D}^p_{en} + \text{sym} \left( \mathbf{F}^p_{en} \mathbf{D}^p_{dis} \mathbf{F}^p_{en}^{-1} \right), \quad \text{with} \quad \text{tr} \mathbf{D}^p_{en} = \text{tr} \mathbf{D}^p_{dis} = 0.
\]  
(2.28)

The right and left polar decompositions of \( \mathbf{F}^p_{en} \) and \( \mathbf{F}^p_{dis} \) are given by
\[
\mathbf{F}^p_{en} = \mathbf{R}^p_{en} \mathbf{U}^p_{en} = \mathbf{V}^p_{en} \mathbf{R}^p_{en}, \quad \mathbf{F}^p_{dis} = \mathbf{R}^p_{dis} \mathbf{U}^p_{dis} = \mathbf{V}^p_{dis} \mathbf{R}^p_{dis},
\]  
(2.29)

where \( \mathbf{R}^p_{en} \) and \( \mathbf{R}^p_{dis} \) are rotations, while \( \mathbf{U}^p_{en}, \mathbf{V}^p_{en}, \mathbf{U}^p_{dis}, \mathbf{V}^p_{dis} \) are symmetric, positive-definite tensors with
\[
\mathbf{U}^p_{en} = \sqrt{\mathbf{F}^p_{en} \mathbf{F}^p_{en}^T}, \quad \mathbf{V}^p_{en} = \sqrt{\mathbf{F}^p_{en} \mathbf{F}^p_{en}^T}, \quad \mathbf{U}^p_{dis} = \sqrt{\mathbf{F}^p_{dis} \mathbf{F}^p_{dis}^T}, \quad \mathbf{V}^p_{dis} = \sqrt{\mathbf{F}^p_{dis} \mathbf{F}^p_{dis}^T}.
\]  
(2.30)

Also, the corresponding right and left Cauchy–Green tensors are given by
\[
\mathbf{C}^p_{en} = \mathbf{U}^{p2}_{en} = \mathbf{F}^p_{en} \mathbf{F}^p_{en}, \quad \mathbf{B}^p_{en} = \mathbf{V}^{p2}_{en} = \mathbf{F}^p_{en} \mathbf{F}^p_{en}^T,
\]  
(2.31)
\[
\mathbf{C}^p_{dis} = \mathbf{U}^{p2}_{dis} = \mathbf{F}^p_{dis} \mathbf{F}^p_{dis}, \quad \mathbf{B}^p_{dis} = \mathbf{V}^{p2}_{dis} = \mathbf{F}^p_{dis} \mathbf{F}^p_{dis}^T.
\]  
(2.32)

3. Frame-indifference

Changes in frame (observer) are smooth time-dependent rigid transformations of the Euclidean space through which the body moves. We require that the theory be invariant under such transformations, and hence under transformations of the form
\[
\mathbf{X}^* (\mathbf{X}, t) = \mathbf{Q}(t) (\mathbf{X}, t) - \mathbf{o} + \mathbf{y}(t),
\]  
(3.1)

with \( \mathbf{Q}(t) \) a rotation (proper-orthogonal tensor), \( \mathbf{y}(t) \) a point at each \( t \), and \( \mathbf{o} \) a fixed origin. Then, under a change in frame, the deformation gradient transforms according to
\[
\mathbf{F}^* = \mathbf{QF},
\]  
(3.2)

and hence
\[
\mathbf{C}^* = \mathbf{C} \quad \text{that is,} \quad \mathbf{C} \text{ is invariant};
\]  
(3.3)

also \( \mathbf{F}^* = \mathbf{QF} + \dot{\mathbf{Q}} \mathbf{F} \), and by (2.1),
\[
\mathbf{L}^* = \mathbf{QLQ}^{-1} + \dot{\mathbf{Q}} \mathbf{Q}^{-1}.
\]  
(3.4)
Thus,
\[ D^* = QDQ^T, \quad W^* = QWQ^T + QQ^T. \]
Moreover, \( (F^p F^p)^T = Q(F^p F^p) \), and therefore, since observers view only the deformed configuration,
\[ F^s = QF^s, \quad F^p = F^p, \]
and thus \( F^p \) is invariant. \( (3.5) \)

Hence, by \( (2.5) \),
\[ L^* = QL^*Q^T + QQ^T, \]
and
\[ L^p, \quad D^p, \quad \text{and} \quad W^p \] are invariant. \( (3.8) \)

Further, by \( (2.10) \),
\[ F^e = R^eU^e \rightarrow QF^e = QR^eU^e, \]
\[ F^e = V^eR^e \rightarrow QF^e = QV^eQ^TQR^e, \]
and we may conclude from the uniqueness of the polar decomposition that
\[ R^e = QR^e, \quad V^e = QV^eQ^T, \quad U^e = U^e. \]
Hence, from \( (2.12) \), \( B^e \) and \( C^e \) transform as
\[ B^e = QB^eQ^T \quad \text{and} \quad C^e = C^e; \]
\( (3.10) \)

Further, since \( F^p \) is invariant,
\[ B^p \quad \text{and} \quad C^p \] are also invariant. \( (3.11) \)

Finally, since \( F^p \) is invariant under a change in frame,
\[ F^p_{\text{en}} \quad \text{and} \quad F^p_{\text{dis}} \] are invariant,
as are all of the following quantities based on \( F^p_{\text{en}} \) and \( F^p_{\text{dis}} \):
\[ U^p_{\text{en}}, \quad C^p_{\text{en}}, \quad V^p_{\text{en}}, \quad B^p_{\text{en}}, \quad R^p_{\text{en}}, \quad U^p_{\text{dis}}, \quad C^p_{\text{dis}}, \quad V^p_{\text{dis}}, \quad B^p_{\text{dis}}, \quad R^p_{\text{dis}}. \]

4. Development of the theory based on the principle of virtual power

Following Gurtin and Anand (2005), the theory presented here is based on the belief that

- the power expended by each independent “rate-like” kinematical descriptor be expressible in terms of an associated force system consistent with its own balance.

However, the basic “rate-like” descriptors, namely, \( v, L^s, D^p_{\text{en}}, \) and \( D^p_{\text{dis}} \) are not independent, since by \( (2.18) \) and \( (2.28) \) they are constrained by
\[ \text{grad} \; v = L^s + F^p D^p F^p^{-1}, \quad \text{tr} D^p = 0, \]
\( (4.1) \)
and
\[ D^p = D^p_{\text{en}} + \text{sym} \left( F^p_{\text{en}} D^p_{\text{dis}} F^p_{\text{en}}^{-1} \right), \quad \text{tr} D^p_{\text{en}} = \text{tr} D^p_{\text{dis}} = 0, \]
\( (4.2) \)
and it is not apparent what forms the associated force balances should take. It is in such situations that the strength of the principle of virtual power becomes apparent, since the principle of virtual power automatically determines the underlying force balances.
We denote by $\mathcal{P}_t$ an arbitrary part (subregion) of the deformed body with $n$ the outward unit normal on the boundary $\partial \mathcal{P}_t$ of $\mathcal{P}_t$. The power expended on $\mathcal{P}_t$ by material or bodies exterior to $\mathcal{P}_t$ results from a macroscopic surface traction $t(n)$, measured per unit area in the deformed body, and a macroscopic body force $b$, measured per unit volume in the deformed body, each of whose working accompanies the macroscopic motion of the body; the body force $b$ presumed to account for inertia; that is, granted the underlying frame is inertial,

$$b = b_0 - \rho \dot{v}, \tag{4.3}$$

with $b_0$ the noninertial body force and $\rho$ the mass density in the deformed body. We therefore write the external power as

$$W_{\text{ext}}(\mathcal{P}_t) = \int_{\partial \mathcal{P}_t} t(n) \cdot \mathbf{v} \, da + \int_{\mathcal{P}_t} b \cdot \mathbf{v} \, dv. \tag{4.4}$$

We assume that power is expended internally by an elastic stress $S^e$ power-conjugate to $L^e$, a plastic stress $T^p$ power-conjugate to $D^p$, an energetic stress $S_{\text{en}}$ conjugate to $D^p_{\text{en}}$, and a dissipative stress $T_{\text{dis}}$ conjugate to $\text{sym}(F^p_{\text{en}} D^p_{\text{dis}} F^p_{\text{en}}^{-1})$. Since $D^p$, $D^p_{\text{en}}$, and $\text{sym}(F^p_{\text{en}} D^p_{\text{dis}} F^p_{\text{en}}^{-1})$ are symmetric and deviatoric, we assume that $T^p$, $S_{\text{en}}$, and $T_{\text{dis}}$ are symmetric and deviatoric. We write the internal power as

$$W_{\text{int}}(\mathcal{P}_t) = \int_{\mathcal{P}_t} \left( S^e : L^e + J^{-1} (T^p : D^p + S_{\text{en}} : D^p_{\text{en}} + T_{\text{dis}} : \text{sym}(F^p_{\text{en}} D^p_{\text{dis}} F^p_{\text{en}}^{-1})) \right) \, dv,$$

$$= \int_{\mathcal{P}_t} \left( S^e : L^e + J^{-1} (T^p : D^p + S_{\text{en}} : D^p_{\text{en}} + \text{sym}(F^p_{\text{en}} T_{\text{dis}} F^p_{\text{en}}^{-T}) : D^p) \right) \, dv. \tag{4.5}$$

The term $J^{-1}$ arises because $T^p : D^p + S_{\text{en}} : D^p_{\text{en}} + T_{\text{dis}} : \text{sym}(F^p_{\text{en}} D^p_{\text{dis}} F^p_{\text{en}}^{-1})$ is measured per unit volume in the structural space, but the integration is carried out within the deformed body. Defining a new stress measure

$$S_{\text{dis}} \overset{\text{def}}{=} \text{sym}(F^p_{\text{en}} T_{\text{dis}} F^p_{\text{en}}^{-T}),$$

the internal power may more compactly be written as

$$W_{\text{int}}(\mathcal{P}_t) = \int_{\mathcal{P}_t} \left( S^e : L^e + J^{-1} (T^p : D^p + S_{\text{en}} : D^p_{\text{en}} + S_{\text{dis}} : D^p_{\text{dis}}) \right) \, dv. \tag{4.6}$$

### 4.1. Principle of virtual power

Assume that, at some arbitrarily chosen but fixed time, the fields $\chi$, $F^e$, $F^p_{\text{en}}$ (and hence $F$, $F^p$, $F^p_{\text{dis}}$) are known, and consider the fields $v$, $L^e$, $D^p$, $D^p_{\text{en}}$, and $D^p_{\text{dis}}$ as virtual velocities to be specified independently in a manner consistent with the constraints (4.1) and (4.2). That is, denoting the virtual fields by $\tilde{v}$, $L^e$, $D^p$, $D^p_{\text{en}}$, and $D^p_{\text{dis}}$ to differentiate them from fields associated with the actual evolution of the body, we require that

$$\text{grad} \, \tilde{v} = L^e + F^e D^p F^p^{-1}, \quad \text{tr} \, D^p = 0, \tag{4.7}$$

and

$$D^p = D^p_{\text{en}} + \text{sym}(F^p_{\text{en}} D^p_{\text{dis}} F^p_{\text{en}}^{-1}). \quad \text{tr} \, D^p_{\text{en}} = \text{tr} \, D^p_{\text{dis}} = 0. \tag{4.8}$$

More specifically, we define a generalized virtual velocity to be a list

$$\nu = (\tilde{v}, L^e, D^p, D^p_{\text{en}}, D^p_{\text{dis}}),$$

consistent with (4.7) and (4.8). Then, writing

$$W_{\text{ext}}(\mathcal{P}_t, \nu) = \int_{\partial \mathcal{P}_t} t(n) \cdot \tilde{v} \, da + \int_{\mathcal{P}_t} b \cdot \tilde{v} \, dv,$$

$$W_{\text{int}}(\mathcal{P}_t, \nu) = \int_{\mathcal{P}_t} \left( S^e : L^e + J^{-1} (T^p : D^p + S_{\text{en}} : D^p_{\text{en}} + S_{\text{dis}} : D^p_{\text{dis}}) \right) \, dv, \tag{4.9}$$
respectively, for the external and internal expenditures of virtual power, the principle of virtual power is the requirement that the external and internal powers be balanced. That is, given any part $\mathcal{P}$,\
\[ W_{\text{ext}}(\mathcal{P}, \nu) = W_{\text{int}}(\mathcal{P}, \nu) \quad \text{for all generalized virtual velocities } \nu. \quad (4.10) \]

4.1. Consequences of frame-indifference

We assume that the internal power $W_{\text{int}}(\mathcal{P}, \nu)$ is invariant under a change in frame and that the virtual fields transform in a manner identical to their nonvirtual counterparts. Then given a change in frame, invariance of the internal power requires that
\[ W_{\text{int}}(\mathcal{P}', \nu^*) = W_{\text{int}}(\mathcal{P}, \nu), \quad (4.11) \]
where $\mathcal{P}'$ and $W_{\text{int}}(\mathcal{P}', \nu^*)$ represent the region and the internal power in the new frame. In the new frame $\mathcal{P}'$, transforms rigidly to a region $\mathcal{P}$, the stresses $\mathbf{S}, \mathbf{T}', \mathbf{S}_e, \text{ and } \mathbf{S}_{\text{dis}}$ transform to $\mathbf{S}', \mathbf{T}', \mathbf{S}_e$, and $\mathbf{S}_{\text{dis}}$, while $\mathbf{L}^e$ transforms to
\[ \mathbf{L}^e = \mathbf{Q} \mathbf{L}' \mathbf{Q}^T + \mathbf{Q} \mathbf{Q}^T, \]
and $\mathbf{D}$, $\mathbf{D}_{\text{en}}$, and $\mathbf{D}_{\text{dis}}$ are invariant. Thus, under a change in frame $W_{\text{int}}(\mathcal{P}, \nu)$ transforms to
\[ W_{\text{int}}(\mathcal{P}', \nu^* \mid \nu) = W_{\text{int}}(\mathcal{P}, \nu) \]
\[ = \int_{\mathcal{P}} (\mathbf{S}^e : (\mathbf{Q} \mathbf{L}' \mathbf{Q}^T + \mathbf{Q} \mathbf{Q}^T) + J^{-1} (\mathbf{T}^p : \mathbf{D}^p + \mathbf{S}_e : \mathbf{D}_{\text{en}} + \mathbf{S}_{\text{dis}} : \mathbf{D}_{\text{dis}})) \, d\nu \]
\[ = \int_{\mathcal{P}} (\mathbf{Q}^T \mathbf{S}^e : \mathbf{Q}) : \mathbf{L}^e + \mathbf{S}^e : (\mathbf{Q} \mathbf{Q}^T) + J^{-1} (\mathbf{T}^p : \mathbf{D}^p + \mathbf{S}_e : \mathbf{D}_{\text{en}} + \mathbf{S}_{\text{dis}} : \mathbf{D}_{\text{dis}})) \, d\nu. \]
Then (4.11) implies that
\[ \int_{\mathcal{P}} (\mathbf{Q}^T \mathbf{S}^e : \mathbf{Q}) : \mathbf{L}^e + \mathbf{S}^e : (\mathbf{Q} \mathbf{Q}^T) + J^{-1} (\mathbf{T}^p : \mathbf{D}^p + \mathbf{S}_e : \mathbf{D}_{\text{en}} + \mathbf{S}_{\text{dis}} : \mathbf{D}_{\text{dis}}) \, d\nu \]
\[ = \int_{\mathcal{P}} (\mathbf{S}^e : \mathbf{L}^e + J^{-1} (\mathbf{T}^p : \mathbf{D}^p + \mathbf{S}_e : \mathbf{D}_{\text{en}} + \mathbf{S}_{\text{dis}} : \mathbf{D}_{\text{dis}})) \, d\nu, \quad (4.12) \]
or equivalently, since the part $\mathcal{P}$ is arbitrary,
\[ (\mathbf{Q}^T \mathbf{S}^e : \mathbf{Q}) : \mathbf{L}^e + \mathbf{S}^e : (\mathbf{Q} \mathbf{Q}^T) + J^{-1} (\mathbf{T}^p : \mathbf{D}^p + \mathbf{S}_e : \mathbf{D}_{\text{en}} + \mathbf{S}_{\text{dis}} : \mathbf{D}_{\text{dis}}) \]
\[ = \mathbf{S}^e : \mathbf{L}^e + J^{-1} (\mathbf{T}^p : \mathbf{D}^p + \mathbf{S}_e : \mathbf{D}_{\text{en}} + \mathbf{S}_{\text{dis}} : \mathbf{D}_{\text{dis}}). \quad (4.13) \]
Also, since the change in frame is arbitrary, if we choose it such that $\mathbf{Q} = \mathbf{0}$, we find from (4.13) that
\[ ((\mathbf{Q}^T \mathbf{S}^e : \mathbf{Q}) - \mathbf{S}^e) : \mathbf{L}^e + J^{-1} (\mathbf{T}^p - \mathbf{T}^p') : \mathbf{D}^p + (\mathbf{S}_e - \mathbf{S}_e) : \mathbf{D}_{\text{en}} + \mathbf{S}_{\text{dis}} : \mathbf{D}_{\text{dis}}) = 0. \]
Since this must hold for all $\mathbf{L}^e$, $\mathbf{D}$, $\mathbf{D}_{\text{en}}$, and $\mathbf{D}_{\text{dis}}$, we find that the stress $\mathbf{S}^e$ transforms according to
\[ \mathbf{S}^e = \mathbf{Q} \mathbf{S}^e \mathbf{Q}^T, \quad (4.14) \]
while $\mathbf{T}^p$, $\mathbf{S}_e$, and $\mathbf{S}_{\text{dis}}$ are invariant
\[ \mathbf{T}^p = \mathbf{T}^p, \quad \mathbf{S}_e = \mathbf{S}_e, \quad \mathbf{S}_{\text{dis}} = \mathbf{S}_{\text{dis}}. \quad (4.15) \]
Next, if we assume that $\mathbf{Q} = \mathbf{1}$ at the time in question, and that $\mathbf{Q}$ is an arbitrary skew tensor, we find from (4.13), using (4.14), (4.15) that
\[ \mathbf{S}^e : \mathbf{Q} = \mathbf{0}, \]
or that the tensor $\mathbf{S}^e$ is symmetric,
\[ \mathbf{S}^e = \mathbf{S}^e^T. \quad (4.16) \]
Thus, the elastic stress $\mathbf{S}^e$ is frame-indifferent and symmetric.
4.1.2. Macroscopic force balance

In applying the virtual balance (4.10) we are at liberty to choose any \( \mathbf{v'} \) consistent with the constraints (4.7) and (4.8). Consider a generalized virtual velocity with \( \mathbf{v} \) for which \( \mathbf{v'} \) is arbitrary, and \( \mathbf{D}^p = \mathbf{D}^p_{\text{en}} = \mathbf{D}^p_{\text{dis}} \equiv 0 \), so that

\[
\mathbf{L}^e = \text{grad} \mathbf{v}.
\]  

(4.17)

For this choice of \( \mathbf{v'} \), (4.10) yields

\[
\int_{\partial \mathcal{P}_t} \mathbf{t}(\mathbf{n}) \cdot \mathbf{v} \, d\mathbf{a} + \int_{\mathcal{P}_t} \mathbf{b} \cdot \mathbf{v} \, d\mathbf{v} = \int_{\mathcal{P}_t} \mathbf{S}' : \text{grad} \mathbf{v} = 0.
\]  

(4.18)

Further, using divergence theorem, (4.18) yields

\[
\int_{\partial \mathcal{P}_t} (\mathbf{t}(\mathbf{n}) - \mathbf{S}' \mathbf{n}) \cdot \mathbf{v} \, d\mathbf{a} + \int_{\mathcal{P}_t} (\text{div} \mathbf{S}' + \mathbf{b}) \cdot \mathbf{v} \, d\mathbf{v} = 0.
\]  

(4.19)

Since (4.19) must hold for all \( \mathcal{P}_t \) and all \( \mathbf{v} \), standard variational arguments yield the traction condition

\[
\mathbf{t}(\mathbf{n}) = \mathbf{S}' \mathbf{n},
\]  

(4.20)

and the local force balance

\[
\text{div} \mathbf{S}' + \mathbf{b} = 0.
\]  

(4.21)

This traction condition and force balance and the symmetry and frame-indifference of \( \mathbf{S}' \) are classical conditions satisfied by the Cauchy stress \( \mathbf{T} \), an observation that allows us to write

\[
\mathbf{T} \equiv \mathbf{S}^e = \mathbf{F}_e \mathbf{T} \mathbf{F}_e^T,
\]  

(4.22)

and to view

\[
\mathbf{T} = \mathbf{T}^T
\]  

(4.23)

as the standard macroscopic Cauchy stress and (4.21) as the local macroscopic force balance. Since we are working in an inertial frame, so that (4.3) is satisfied, (4.21) reduces to the local balance law for linear momentum,

\[
\text{div} \mathbf{T} + \mathbf{b}_0 = \rho \mathbf{v},
\]  

(4.24)

with \( \mathbf{b}_0 \) the noninertial body force.

4.1.3. Microscopic force balances

To discuss the microscopic counterparts of the macroscopic force balance, first consider a generalized virtual velocity with

\[
\mathbf{v} \equiv 0 \quad \text{and} \quad \mathbf{D}^p_{\text{dis}} \equiv 0;
\]  

(4.25)

also, choose the virtual field \( \mathbf{D}^p \) arbitrarily, and let

\[
\mathbf{L}^e = -\mathbf{F}^T \mathbf{D}^p \mathbf{F}^{-1} \quad \text{and} \quad \mathbf{D}^p_{\text{en}} = \mathbf{D}^p,
\]  

(4.26)

consistent with (4.7) and (4.8). Thus

\[
\mathbf{T} : \mathbf{L}^e = -(\mathbf{F}^T \mathbf{T} \mathbf{F}^{-T}) : \mathbf{D}^p.
\]  

(4.27)

Next, define an elastic Mandel stress by

\[
\mathbf{M}^e \equiv \mathbf{J} \mathbf{F}^T \mathbf{T} \mathbf{F}^{-T},
\]  

(4.28)

which in general is not symmetric. Then, on account of our choice (4.25), the external power vanishes, so that, by (4.10), the internal power must also vanish, and satisfy
\[ W_{\text{int}}(\mathcal{P},v) = \int_{\mathcal{P}} (T^p + S_{\text{en}} - M^e) : D^p \, dv = 0. \]

Since this must be satisfied for all \( \mathcal{P} \) and all symmetric and deviatoric tensors \( D^p \), a standard argument yields the first microforce balance

\[ \text{sym}_0 M^e - S_{\text{en}} = T^p. \] (4.29)

Next consider a generalized virtual velocity with

\[ \tilde{v} \equiv 0 \quad \text{and} \quad \tilde{D}^p \equiv 0; \] (4.30)

also, choose the virtual field \( \tilde{D}^p_{\text{dis}} \) arbitrarily, and let

\[ \tilde{D}^p_{\text{en}} = -\text{sym} \left( F^p_{\text{en}} D^p_{\text{dis}} F^{p-1}_{\text{en}} \right) \] (4.31)

consistent with (4.8). Thus

\[ S_{\text{en}} : \tilde{D}^p_{\text{en}} = -\left( F^p_{\text{en}} F^{p-1}_{\text{en}} \right) : \tilde{D}^p_{\text{dis}}. \] (4.32)

Next, define a new stress measure by

\[ M^p_{\text{en}} = F^p_{\text{en}} S_{\text{en}} F^{p-1}_{\text{en}}, \] (4.33)

which in general is also not symmetric. For lack of better terminology, we call \( M^p_{\text{en}} \) a plastic Mandel stress. Then, on account of our choice (4.30), the external power vanishes, so that, by (4.10), the internal power must also vanish, and satisfy

\[ W_{\text{int}}(\mathcal{P},v) = \int_{\mathcal{P}} (S_{\text{dis}} - M^p_{\text{en}}) : D^p_{\text{dis}} \, dv = 0. \]

Since this must be satisfied for all \( \mathcal{P} \), and all symmetric and deviatoric tensors \( D^p_{\text{dis}} \), we obtain the second microforce balance

\[ \text{sym}_0 M^p_{\text{en}} = S_{\text{dis}}. \] (4.34)

5. Free-energy imbalance

We consider a purely mechanical theory based on a second law requiring that the temporal increase in free energy of any part \( \mathcal{P} \) be less than or equal to the power expended on \( \mathcal{P} \). The second law therefore takes the form of a dissipation inequality

\[ \int_{\mathcal{P}} \psi J^{-1} \, dv \leq W_{\text{ext}}(\mathcal{P}) = W_{\text{int}}(\mathcal{P}), \] (5.1)

with \( \psi \) the free energy, measured per unit volume in the structural space.

Since \( J^{-1} \, dv \) with \( J = \det F \) represents the volume measure in the reference configuration, and since \( \mathcal{P} \) deforms with the body,

\[ \int_{\mathcal{P}} \psi J^{-1} \, dv = \int_{\mathcal{P}} \psi J^{-1} \, dv. \]

Thus, since \( \mathcal{P} \) is arbitrary, we may use (4.6), (4.22), and (4.23) to localize (5.1); the result is the local free-energy imbalance
\[
\dot{\psi} - J^T : D^e - T^0 : D^p - S_{en} : D^p_{en} - S_{dis} : D^p_{dis} \leq 0.
\]  \hfill (5.2)

We introduce a new stress measure
\[
T^e \overset{\text{def}}{=} J F^{e^{-1}} T F^{e^{-T}},
\]  \hfill (5.3)

which is symmetric, since \(T\) is symmetric; \(T^e\) represents a second Piola stress with respect to the intermediate structural space. Note that by using the definition (5.3), the Mandel stress defined in (4.28) is related to the stress measure \(T^e\) by
\[
M^e = C^e T^e.
\]  \hfill (5.4)

Next, differentiating (2.12), results in the following expression for the rate of change of \(C^e\),
\[
\dot{C}^e = F^e \dot{F}^e + F^e \dot{F}^e.
\]

Hence, since \(T^e\) is symmetric,
\[
T^e : \dot{C}^e = 2T^e : (F^e \dot{F}^e) = 2(F^e T^e F^{e^T}) : (F^e F^{e^T}) = 2(F^e T^e F^{e^T}) : D^e,
\]
or, using (5.3) we obtain
\[
J^T : D^e = \frac{1}{2} T^e : \dot{C}^e.
\]  \hfill (5.5)

Use of (5.5) in (5.2) allows us to express the free-energy imbalance in an alternate convenient form as
\[
\dot{\psi} - \frac{1}{2} T^e : \dot{C}^e - T^0 : D^p - S_{en} : D^p_{en} - S_{dis} : D^p_{dis} \leq 0.
\]  \hfill (5.6)

Finally, we note that \(\dot{\psi}\) is invariant under a change in frame since it is a scalar field, and on account of the transformation rules discussed in Section 3, the transformation rules (4.14) and (4.15), and the definitions (4.28), (5.3), (4.33), and (5.4), the fields
\[
C^e, \quad C^e_{en}, \quad B^p_{en}, \quad D^p, \quad D^p_{en}, \quad D^p_{dis}, \quad T^e, \quad T^0, \quad S_{en}, \quad S_{dis}, \quad M^e, \quad \text{and} \quad M^e_{en}
\]  \hfill (5.7)

are also invariant.

6. Constitutive theory

To account for the classical notion of isotropic strain hardening we introduce a positive-valued scalar internal state-variable \(S\), which has dimensions of stress. We refer to \(S\) as the isotropic deformation resistance. Since \(S\) is a scalar field, it is invariant under a change in frame.

Next, guided by the free-energy imbalance (5.6), and by experience with previous constitutive theories, we assume the following special set of constitutive equations:
\[
\begin{align*}
\psi &= \tilde{\psi}^{(e)}(C^e) + \tilde{\psi}^{(p)}(B^p_{en}), \\
T^e &= T^e(C^e), \\
S_{en} &= \tilde{S}_{en}(B^p_{en}), \\
T^0 &= \tilde{T}^0(D^p, S), \\
S_{dis} &= \tilde{S}_{dis}(D^p_{dis}, \tilde{D}^p, S), \\
\dot{S} &= h(\tilde{D}^p, S),
\end{align*}
\]  \hfill (6.1)

where
\[
\tilde{D}^p \overset{\text{def}}{=} |D^p|
\]  \hfill (6.2)

is the scalar flow rate corresponding to \(D^p\). Note that since \(\psi\) is the free energy per unit volume of the structural space, \(\tilde{\psi}^{(e)}\) is chosen to depend on \(C^e\) and \(\tilde{\psi}^{(p)}\) on \(B^p_{en}\), because \(C^e\) and \(B^p_{en}\) are squared stretch-like measures associated with the intermediate structural space. Also note that we have introduced a
possible dependence of $S_{\text{dis}}$ on $d^p$ because for some materials we anticipate that $D_{\text{dis}}^p$ (and thereby $S_{\text{dis}}$) may be constrained by the magnitude of $d^p$.

The energy $\psi^{(e)}$ represents the standard energy associated with intermolecular interactions, and the energy $\psi^{(p)}$ is a "defect-energy" associated with plastic deformation. For metallic materials $\psi^{(p)}$ may be attributed to an energy stored in the complex local microstructural defect state, which typically includes dislocations, sub-grain boundaries, and local incompatibilities at second-phase particles and grain boundaries. At the macroscopic continuum level, the "defect-energy" $\psi^{(p)}$ leads to the development and evolution of the energetic stress $S_{\text{en}}$, and this allows one to phenomenologically account for strain-hardening phenomena commonly called kinematic hardening. Isotropic hardening is modeled by the evolution (6.1)_{6} of the isotropic deformation resistance $S$.

Finally, note that on account of the transformation rules listed in the paragraph containing (5.7) and since $S$ is also invariant, the constitutive equations (6.1) are frame-indifferent.

6.1. Thermodynamic restrictions

From (6.1)$_{1}$

$$\dot{\psi} = \frac{\partial \psi^{(e)}(C^e)}{\partial C^e} : \dot{C}^e + \frac{\partial \psi^{(p)}(B^p_{\text{en}})}{\partial B^p_{\text{en}}} : \dot{B}^p_{\text{en}},$$

and, using$_{8}$

$$\dot{B}^p_{\text{en}} = D^p_{\text{en}} B^p_{\text{en}} + B^p_{\text{en}} D^p_{\text{en}};$$

and the symmetry of $B^p_{\text{en}}$ and $\frac{\partial \psi^{(p)}}{\partial B^p_{\text{en}}}$,

$$\frac{\partial \psi^{(p)}}{\partial B^p_{\text{en}}} : B^p_{\text{en}} = \frac{\partial \psi^{(p)}(B^p_{\text{en}})}{\partial B^p_{\text{en}}} : (D^p_{\text{en}} B^p_{\text{en}} + B^p_{\text{en}} D^p_{\text{en}})$$

$$= 2 \left( \frac{\partial \psi^{(p)}(B^p_{\text{en}})}{\partial B^p_{\text{en}}} B^p_{\text{en}} \right) : D^p_{\text{en}}.$$  

Hence, satisfaction of the free-energy imbalance (5.6) requires that the constitutive equations (6.1) satisfy

$$\left( \frac{1}{2} T^e(C^e) - \frac{\partial \psi^{(e)}(C^e)}{\partial C^e} : \dot{C}^e + \left( S_{\text{en}}(B^p_{\text{en}}) - 2 \left( \frac{\partial \psi^{(p)}(B^p_{\text{en}})}{\partial B^p_{\text{en}}} B^p_{\text{en}} \right) \right) : D^p_{\text{en}} + T^p(D^p, S) : D^p + S_{\text{dis}}(D^p_{\text{dis}}, d^p, S) : D^p_{\text{dis}} \right) \geq 0,$$

and hold for all arguments in the domains of the constitutive functions, and in all motions of the body. Thus, sufficient conditions that the constitutive equations satisfy the free-energy imbalance are that$_{9}$

(i) the free-energy determine the stresses $T^e$ and $S_{\text{en}}$ via the stress relations

$$T^e(C^e) = 2 \frac{\partial \psi^{(e)}(C^e)}{\partial C^e},$$

$$S_{\text{en}}(B^p_{\text{en}}) = 2 \left( \frac{\partial \psi^{(p)}(B^p_{\text{en}})}{\partial B^p_{\text{en}}} B^p_{\text{en}} \right).$$

(ii) the plastic distortion-rates $D^p$ and $D^p_{\text{dis}}$ satisfy the dissipation inequality

$$T^p : D^p + S_{\text{dis}} : D^p_{\text{dis}} \geq 0.$$  

We assume henceforth that (6.5) and (6.6) hold in all motions of the body. We assume further that the material is strictly dissipative in the sense that

$_{8}$ Recall that $W^p_{\text{en}} = 0$, cf., (2.26).

$_{9}$ We content ourselves with constitutive equations that are only sufficient, but generally not necessary for compatibility with thermodynamics.
T^p : D^p > 0 \text{ whenever } D^p \neq 0, \quad (6.8)
and
S_{\text{dis}} : D_{\text{dis}}^p > 0 \text{ whenever } D_{\text{dis}}^p \neq 0. \quad (6.9)

Some remarks: It is possible to enhance the theory by introducing another positive-valued dimensionless strain-like scalar internal variable $\kappa$ which contributes an additional defect energy $\psi_{iso}^{(p)}(\kappa)$, which evolves as $\kappa$ evolves according to an evolution equation of the type $\dot{\kappa} = h(d^p, \kappa)$. This will lead to a scalar internal stress $K(\kappa) \equiv \partial \psi_{iso}^{(p)}(\kappa)/\partial \kappa$, and the dissipation inequality (6.7) will be modified as

$$T^p : D^p + S_{\text{dis}} : D_{\text{dis}}^p - K\kappa \geq 0. \quad (6.10)$$

Although a term such as $\psi_{iso}^{(p)}(\kappa)$ might be needed for a more complete characterization of the “stored energy of cold work,” and a more precise determination of dissipation and thereby heat generation (cf., e.g., Rosakis et al., 2000; Ristinmaa et al., 2007), here, since our interest is in an isothermal theory, we refrain from introducing such additional complexity. However, we do wish to emphasize that our internal variable $S$, which represents an isotropic deformation resistance to plastic flow, is in no sense introduced as being derived from a defect energy $\psi_{iso}^{(p)}(\kappa)$ – it is introduced on a purely pragmatic phenomenological basis. Note that Ristinmaa et al. (2007) consider only “isotropic” hardening theories, while the theory under consideration here is for combined isotropic and kinematic hardening, and the term $\psi_{iso}^{(p)}(\mathbf{B}_{en}^p)$ already accounts for some aspects of the “stored energy of cold work.”

Vladimirov et al. (2008) have considered a combined isotropic and kinematic-hardening theory with an additional defect energy $\psi_{iso}^{(p)}(\kappa)$, and write their corresponding stress as $\partial \psi_{iso}^{(p)}(\kappa)/\partial \kappa = (-R(\kappa)) \equiv K(\kappa)$. They associate the stress $(-R(\kappa))$ as the strain-hardening contribution to the isotropic deformation resistance, and take $\dot{\kappa} \equiv (\sqrt{2}/3)d^p$, so that $\kappa$ represents an accumulated plastic strain. It is our opinion that for materials which show strong isotropic strain hardening, the theory of Reese and co-workers will grossly underestimate the amount of dissipation. This is the major reason why we do not follow the approach of Reese and co-workers, but directly introduce our stress-like internal variable $S$ to represent isotropic strain hardening.

6.2. Isotropy

The following definitions help to make precise our notion of an isotropic material (cf., Anand and Gurtin, 2003):

(i) Orth $= \text{the group of all rotations (the proper-orthogonal group)}$;
(ii) the symmetry group $\mathcal{G}_s$ is the group of all rotations of the reference configuration that leaves the response of the material unaltered;
(iii) the symmetry group $\mathcal{G}_s$ at each time $t$, is the group of all rotations of the structural space that leaves the response of the material unaltered;
(iv) the symmetry group $\mathcal{G}_{ss}$ at each time $t$, is the group of all rotations of the substructural space that leaves the response of the material unaltered.

Note that there is a slight difficulty in attaching a “physical meaning” to a rotation in $\mathcal{G}_{ss}$. Since the substructural space itself is a mathematical construct defined as the range of the linear map $\mathbf{F}_{\text{dis}}^p$, a rotation $\mathbf{Q} \in \mathcal{G}_{ss}$ should be considered as a rotation in a “thought experiment” involving the substructural space; such a rotation may never be “physically attainable.” We now discuss the manner in which the basic fields transform under such transformations, granted the physically natural requirement of invariance of the internal power or equivalently, the requirement that

$$T^p : \mathbf{C}, \quad T^p : D^p, \quad S_{\text{en}} : D_{\text{en}}^p, \quad \text{and } S_{\text{dis}} : D_{\text{dis}}^p \text{ be invariant.} \quad (6.11)$$

6.2.1. Isotropy of the reference space

Let $\mathbf{Q}$ be a time-independent rotation of the reference configuration. Then $\mathbf{F} \rightarrow \mathbf{FQ}$, and hence
\[ \mathbf{F}^p \to \mathbf{F}^p \mathbf{Q}, \quad \mathbf{F}^p_{\text{dis}} \to \mathbf{F}^p_{\text{dis}} \mathbf{Q}, \quad \mathbf{F}^e_{\text{en}} \to \mathbf{F}^e_{\text{en}} \mathbf{Q}, \quad \mathbf{F}^e_{\text{dis}} \to \mathbf{F}^e_{\text{dis}} \mathbf{Q}, \text{ and } \mathbf{F}^p \text{ and } \mathbf{F}^p_{\text{dis}} \text{ are invariant, and hence } \mathbf{C}^e \text{ and } \mathbf{C}^p_{\text{en}} \text{ are invariant}, \]

so that, by (2.5), (2.12), and (2.21),

\[ \mathbf{C}^e, \mathbf{D}^p, \mathbf{D}^p_{\text{en}}, \text{ and } \mathbf{D}^p_{\text{dis}} \text{ are invariant.} \]

We may therefore use (6.11) to conclude that

\[ \mathbf{T}^e, \mathbf{T}^p, \mathbf{S}^e_{\text{en}}, \text{ and } \mathbf{S}^e_{\text{dis}} \text{ are invariant.} \]

Thus, with reference to the special constitutive functions (6.1), we note the these functions are not affected by rotations \( \mathbf{Q} \in \mathcal{G}_R \).

6.2.2. Isotropy of the structural space

Next, let \( \mathbf{Q} \), a time-independent rotation of the intermediate structural space, be a symmetry transformation. Then \( \mathbf{F} \) is unaltered by such a rotation, and hence

\[ \mathbf{F}^e \to \mathbf{F}^e \mathbf{Q}, \quad \mathbf{F}^p \to \mathbf{Q}^\top \mathbf{F}^p, \quad \mathbf{F}^p_{\text{en}} \to \mathbf{Q}^\top \mathbf{F}^p_{\text{en}}, \text{ and } \mathbf{F}^p_{\text{dis}} \text{ is invariant}, \]

and also

\[ \mathbf{C}^e \to \mathbf{Q}^\top \mathbf{C}^e \mathbf{Q}, \quad \mathbf{C}^e \to \mathbf{Q}^\top \mathbf{C}^e \mathbf{Q}, \quad \mathbf{D}^p \to \mathbf{Q}^\top \mathbf{D}^p \mathbf{Q}, \quad \mathbf{B}^p_{\text{en}} \to \mathbf{Q}^\top \mathbf{B}^p_{\text{en}} \mathbf{Q}, \]

\[ \mathbf{D}^p_{\text{dis}} \to \mathbf{Q}^\top \mathbf{D}^p_{\text{dis}} \mathbf{Q}, \text{ and } \mathbf{D}^p_{\text{dis}} \text{ is invariant.} \]

Then (6.15) and (6.11) yield the transformation laws

\[ \mathbf{T}^e \to \mathbf{Q}^\top \mathbf{T}^e \mathbf{Q}, \quad \mathbf{T}^p \to \mathbf{Q}^\top \mathbf{T}^p \mathbf{Q}, \quad \mathbf{S}^e_{\text{en}} \to \mathbf{Q}^\top \mathbf{S}^e_{\text{en}} \mathbf{Q}, \text{ and } \mathbf{S}^e_{\text{dis}} \text{ is invariant.} \]

Thus, with reference to the constitutive equations (6.1) we conclude that

\[ \begin{align*}
\tilde{\psi}^{(e)}(\mathbf{C}^e) &= \mathbf{Q}^\top (\mathbf{C}^e) \mathbf{Q}, \\
\tilde{\psi}^{(p)}(\mathbf{B}^p_{\text{en}}) &= \mathbf{Q}^\top (\mathbf{B}^p_{\text{en}}) \mathbf{Q}, \\
\mathbf{Q}^\top \mathbf{T}^e(\mathbf{C}^e) \mathbf{Q} &= \mathbf{T}^e(\mathbf{C}^e) \mathbf{Q}, \\
\mathbf{Q}^\top \mathbf{S}^e_{\text{en}}(\mathbf{B}^p_{\text{en}}) \mathbf{Q} &= \mathbf{S}^e_{\text{en}}(\mathbf{B}^p_{\text{en}}) \mathbf{Q}, \\
\mathbf{Q}^\top \mathbf{T}^p(\mathbf{D}^p, S) \mathbf{Q} &= \mathbf{T}^p(\mathbf{D}^p, S) \mathbf{Q}.
\end{align*} \]

must hold for all rotations \( \mathbf{Q} \) in the symmetry group \( \mathcal{G}_S \) at each time \( t \), while the constitutive functions \( \mathbf{S}^e_{\text{dis}}(\mathbf{D}^e_{\text{dis}}), \mathbf{d}^e(S), \) and \( h(d^e, S) \) are invariant to such rotations.

We refer to the material as one which is structurally isotropic if

\[ \mathcal{G}_S = \text{Orth}^+, \]

so that the response of the material is invariant under arbitrary rotations of the intermediate structural space at each time \( t \). Henceforth, we assume that (6.18) holds. In this case, the response functions \( \tilde{\psi}^{(e)}, \tilde{\psi}^{(p)}, \mathbf{T}^e, \mathbf{S}^e_{\text{en}}, \) and \( \mathbf{T}^p \) must each be isotropic.

6.2.3. Isotropy of the substructural space

Finally, let \( \mathbf{Q} \), a time-independent rotation of the substructural space, be a symmetry transformation. Then \( \mathbf{F}, \mathbf{F}^e, \) and \( \mathbf{F}^p \) are unaltered by such a rotation, and hence

\[ \mathbf{F}^p_{\text{en}} \to \mathbf{F}^p_{\text{en}} \mathbf{Q}, \quad \mathbf{F}^p_{\text{dis}} \to \mathbf{Q}^\top \mathbf{F}^p_{\text{dis}}, \]

and also

\[ \mathbf{D}^p_{\text{dis}} \to \mathbf{Q}^\top \mathbf{D}^p_{\text{dis}} \mathbf{Q}, \text{ and } \mathbf{D}^p_{\text{dis}} \text{ is invariant.} \]

Then (6.20) and (6.11) yield the transformation laws

\[ \mathbf{S}^e_{\text{dis}} \to \mathbf{Q}^\top \mathbf{S}^e_{\text{dis}} \mathbf{Q}, \text{ and } \mathbf{S}^e_{\text{en}} \text{ is invariant.} \]
Thus, with reference to the constitutive equations (6.1) we conclude that
\[ Q^\top S_{\text{dis}}(D^p_{\text{dis}}, d^p, S) Q = S_{\text{dis}}(Q^\top D^p_{\text{dis}} Q, d^p, S) \]  
(6.22)
must hold for all rotations \( Q \) in the symmetry group \( \mathcal{G}_{SS} \) at each time \( t \), while the other constitutive functions are invariant to such rotations.

We refer to the material as one which is substructurally isotropic if
\[ \mathcal{G}_{SS} = \text{Orth}^+ \]  
(6.23)
so that the response of the material is also invariant under arbitrary rotations of the substructural space at each time. Henceforth, we assume that (6.23) also holds at each time, so that \( S_{\text{dis}} \) is an isotropic function.

Thus, by assumption, all the tensorial response functions in (6.1) are isotropic. We confine our attention to materials which may be adequately defined by such isotropic functions, and refer to such materials as isotropic.

6.3. Consequences of isotropy of the free energy

Since \( \bar{\psi}^{(p)}(C^e) \) is an isotropic function of \( C^e \), it has the representation
\[ \bar{\psi}^{(p)}(C^e) = \bar{\psi}^{(p)}(\mathcal{I} C^e), \]  
(6.24)
where
\[ \mathcal{I} C^e = (I_1(C^e), I_2(C^e), I_3(C^e)) \]
is the list of principal invariants of \( C^e \). Thus, from (6.5),
\[ T^e = \mathbf{T}^e(C^e) = 2 \frac{\partial \bar{\psi}^{(p)}(\mathcal{I} C^e)}{\partial C^e}, \]  
(6.25)
and \( \mathbf{T}^e(C^e) \) is an isotropic function of \( C^e \). Then since the Mandel stress is defined by (cf., (5.4))
\[ M^e = C^e T^e, \]
and \( \mathbf{T}^e(C^e) \) is isotropic, we find that \( T^e \) and \( C^e \) commute,
\[ C^e T^e = T^e C^e, \]  
(6.26)
and hence that the elastic Mandel stress \( M^e \) is symmetric.

Further the defect free energy has a representation
\[ \bar{\psi}^{(p)} = \bar{\psi}^{(p)}(\mathcal{I} B_{\text{en}}^p) \text{ where } \mathcal{I} B_{\text{en}}^p = (I_1(B_{\text{en}}^p), I_2(B_{\text{en}}^p), I_3(B_{\text{en}}^p)), \]  
(6.27)
and this yields that
\[ \frac{\partial \bar{\psi}^{(p)}(\mathcal{I} B_{\text{en}}^p)}{\partial B_{\text{en}}^p} B_{\text{en}}^p \]  
(6.28)
is a symmetric tensor, and (6.6) reduces to
\[ S_{\text{en}} = S_{\text{en}}(B_{\text{en}}^p) = 2 \left( \frac{\partial \bar{\psi}^{(p)}(\mathcal{I} B_{\text{en}}^p)}{\partial B_{\text{en}}^p} B_{\text{en}}^p \right)_0, \]  
(6.29)
a symmetric and deviatoric tensor.

Also the plastic Mandel stress defined in (4.33) is
\[ M_{\text{en}}^p = 2 P_{\text{en}}^p \left( \frac{\partial \bar{\psi}^{(p)}(\mathcal{I} B_{\text{en}}^p)}{\partial B_{\text{en}}^p} B_{\text{en}}^p \right)_0 P_{\text{en}}^{p\top}, \]  

The constitutive equation (6.4) yield the
Then, upon using the constitutive relation (6.1) and the first microforce balance (4.29), together with
and since

7. Flow rules

Henceforth, in accord with common terminology we call the symmetric and deviatoric tensor $S_{en}$ a back-stress, and we denote an effective deviatoric Mandel stress by

$$(M^{en}_{en})_0 \overset{\text{def}}{=} M^0_0 - S_{en}. \quad (7.1)$$

Then, upon using the constitutive relation (6.1) and the first microforce balance (4.29), together with

$$(M^{en}_{en})_0 = \mathbf{T}^T(D^p, S). \quad (7.2)$$

Next, on account of the symmetry and deviatoric nature of the plastic stress $M^p_{en}$ (cf., (6.30)), the second microforce balance (4.34) and the constitutive equation (6.4) yield the second flow rule of the theory:

$$M^p_{en} = \mathbf{S}_{\text{dis}}(D^p_{\text{dis}}, D^p, S). \quad (7.3)$$

We now make two major assumptions concerning the plastic flow for isotropic materials:

1. **Codirectionality hypotheses:** Recall

$$d^p = |D^p|,$n and let

$$d^p_{\text{dis}} = |D^p_{\text{dis}}|;
$$

also let

$$\mathbf{N}^p \overset{\text{def}}{=} \frac{D^p}{d^p} \quad \text{and} \quad \mathbf{N}^p_{\text{dis}} \overset{\text{def}}{=} \frac{D^p_{\text{dis}}}{d^p_{\text{dis}}} \quad (7.4)$$

denote the directions of plastic flow whenever $D^p \neq 0$ and $D^p_{\text{dis}} \neq 0$. Then the dissipation inequality (6.7) may be written as

$$(\mathbf{T}^T(d^p, \mathbf{N}^p, S) : \mathbf{N}^p)d^p + (\mathbf{S}_{\text{dis}}(d^p_{\text{dis}}, \mathbf{N}^p_{\text{dis}}, d^p, S) : \mathbf{N}^p_{\text{dis}})d^p_{\text{dis}} \geq 0. \quad (7.5)$$

Guided by (7.5), we assume henceforth that the dissipative flow stress $\mathbf{T}^p$ is parallel to and points in the same direction as $\mathbf{N}^p$ so that

$$\mathbf{T}^p(d^p, \mathbf{N}^p, S) = Y(d^p, \mathbf{N}^p, S)\mathbf{N}^p, \quad (7.6)$$

where

$$Y(d^p, \mathbf{N}^p, S) = \mathbf{T}^p(d^p, \mathbf{N}^p, S) : \mathbf{N}^p \quad (7.7)$$

represents a scalar flow strength of the material.

Similarly we assume that dissipative flow stress $\mathbf{S}_{\text{dis}}$ is parallel to and points in the same direction as $\mathbf{N}^p_{\text{dis}}$ so that

$$\mathbf{S}_{\text{dis}}(d^p_{\text{dis}}, \mathbf{N}^p_{\text{dis}}, d^p, S) = Y_{\text{dis}}(d^p_{\text{dis}}, \mathbf{N}^p_{\text{dis}}, d^p, S)\mathbf{N}^p_{\text{dis}}, \quad (7.8)$$

where

$$Y_{\text{dis}}(d^p_{\text{dis}}, \mathbf{N}^p_{\text{dis}}, d^p, S) = \mathbf{S}_{\text{dis}}(d^p_{\text{dis}}, \mathbf{N}^p_{\text{dis}}, d^p, S) : \mathbf{N}^p_{\text{dis}} \quad (7.9)$$

represents another scalar flow strength of the material.

We refer to the assumptions (7.6) and (7.8) as the codirectionality hypotheses.\(^\text{10}\)

\(^\text{10}\) These assumptions correspond to the classical notion of maximal dissipation in Mises-type theories of metal plasticity.
(2) **Strong isotropy hypotheses:**

We also assume that the scalar flow strength $Y(d^p, N^p, S)$ is independent of the flow direction $N^p$, so that

$$Y(d^p, S).$$

Similarly, we also assume that the scalar flow strength $Y_{dis}(d^p_{dis}, N^p_{dis}, d^p, S)$ is independent of the flow direction $N^p_{dis}$, so that

$$Y_{dis}(d^p_{dis}, d^p, S).$$

We refer to the assumptions (7.10) and (7.11) as the **strong isotropy hypotheses.**

Thus, using (7.10) and (7.6), the flow rule (7.2) reduces to,

$$\left(\mathbf{M}_{eff}^e\right)_0 = Y(d^p, S) \mathbf{N}^p,$$

which immediately gives

$$\mathbf{N}^p = \left(\mathbf{M}_{eff}^e\right)_0 / |\left(\mathbf{M}_{eff}^e\right)_0|,$$

and

$$|\left(\mathbf{M}_{eff}^e\right)_0| = Y(d^p, S).$$

When $|\left(\mathbf{M}_{eff}^e\right)_0|$ and $S$ are known, (7.14) serves as an implicit equation for the scalar flow rate $d^p$.

Next, using (7.11) and (7.8), the flow rule (7.3) reduces to,

$$\mathbf{M}_{en}^p = Y_{dis}(d^p_{dis}, d^p, S) \mathbf{N}^p_{dis},$$

which gives

$$\mathbf{N}^p_{dis} = \mathbf{M}_{en}^p / |\mathbf{M}_{en}^p|,$$

and

$$|\mathbf{M}_{en}^p| = Y_{dis}(d^p_{dis}, d^p, S).$$

When $|\mathbf{M}_{en}^p|$ and $S$ are known, (7.17) serves as an implicit equation for the scalar flow rate $d^p_{dis}$. Finally, using (7.6), (7.10), and (7.14), as well as (7.8), (7.11), and (7.17) the dissipation inequality (6.7) reduces to

$$|\left(\mathbf{M}_{eff}^e\right)_0| d^p + |\mathbf{M}_{en}^p| d^p_{dis} \geq 0.$$  

8. Summary of the constitutive theory

In this section we summarize our constitutive theory.

(1) **Free energy.**

$$\psi = \tilde{\psi}^{(e)}(\mathscr{F}^e) + \tilde{\psi}^{(p)}(\mathscr{F}_{en}^p),$$

where $\mathscr{F}^e$ and $\mathscr{F}_{en}^p$ are the lists of the principal invariants of $\mathbf{C}^e$ and $\mathbf{B}_{en}^p$, respectively.

(2) **Cauchy stress.**

$$\mathbf{T} \equiv \mathbf{f}^{-1}(\mathbf{F}^e \mathbf{F}^{e\top}) \quad \text{where} \quad \mathbf{T}^e = 2 \frac{\partial \tilde{\psi}^{(e)}(\mathscr{F}^e)}{\partial \mathbf{C}^e}.$$

(3) **Mandel stresses. Back-stress.** The elastic Mandel is given by

$$\mathbf{M}^e = \mathbf{C}^e \mathbf{T}^e.$$
and is symmetric. The back-stress is given by
\[
S_{\text{en}} = 2 \left( \frac{\partial \psi^{(p)}}{\partial \mathbf{F}^p_{\text{en}}} \mathbf{B}_{\text{en}}^p \right)_0,
\]
and is symmetric deviatoric. The stress difference
\[
M^e_{\text{eff}} = M^e - S_{\text{en}}
\]
is called an effective elastic Mandel stress. The plastic Mandel stress is given by
\[
M^p_{\text{en}} = 2 \left[ \mathbf{F}^p_{\text{en}} \left( \frac{\partial \psi^{(p)}}{\partial \mathbf{F}^p_{\text{en}}} \right) \mathbf{F}^p_{\text{en}} \right]_0,
\]
and is symmetric and deviatoric.

(4) Flow rules. The evolution equation for \( \mathbf{F}^p \) is
\[
\dot{\mathbf{F}}^p = \mathbf{D}^p \mathbf{F}^p,
\]
with \( \mathbf{D}^p \) given by
\[
\mathbf{D}^p = d^p \mathbf{N}^p, \quad \mathbf{N}^p = \frac{(M^e_{\text{eff}})_0}{|M^e_{\text{eff}}|},
\]
where the scalar flow rate \( d^p \) is obtained by solving the scalar strength relation
\[
|M^e_{\text{eff}}| = Y(d^p, S),
\]
for given \( (M^e_{\text{eff}})_0 \) and \( S \), where \( Y(d^p, S) \) is a rate-dependent flow strength. The evolution equation for \( \mathbf{F}^p_{\text{dis}} \) is
\[
\dot{\mathbf{F}}^p_{\text{dis}} = \mathbf{D}^p_{\text{dis}} \mathbf{F}^p_{\text{dis}},
\]
with \( \mathbf{D}^p_{\text{dis}} \) given by
\[
\mathbf{D}^p_{\text{dis}} = d^p_{\text{dis}} \mathbf{N}^p_{\text{dis}}, \quad \mathbf{N}^p_{\text{dis}} = \frac{M^p_{\text{en}}}{|M^p_{\text{en}}|},
\]
where the scalar flow rate \( d^p_{\text{dis}} \) is obtained by solving the scalar strength relation
\[
|M^p_{\text{en}}| = Y_{\text{dis}}(d^p_{\text{dis}}, d^p, S),
\]
for given \( M^p_{\text{en}}, d^p, \) and \( S \), where \( Y_{\text{dis}}(d^p_{\text{dis}}, d^p, S) \) is another rate-dependent flow strength.

(5) Evolution equation for \( S \).
\[
\dot{S} = h(d^p, S).
\]

The evolution equations for \( \mathbf{F}^p, \mathbf{F}^p_{\text{dis}}, \) and \( S \) need to be accompanied by initial conditions. Typical initial conditions presume that the body is initially (at time \( t = 0, \) say) in a virgin state in the sense that
\[
\mathbf{F}(\mathbf{X}, 0) = \mathbf{F}^p(\mathbf{X}, 0) = \mathbf{F}^p_{\text{dis}}(\mathbf{X}, 0) = 1, \quad S(\mathbf{X}, 0) = S_0 (= \text{constant}),
\]
so that by \( \mathbf{F} = \mathbf{F}^p \mathbf{F}^p_{\text{dis}} \) and \( \mathbf{F}^p = \mathbf{F}^p_{\text{en}} \mathbf{F}^p_{\text{dis}} \) we also have \( \mathbf{F}(\mathbf{X}, 0) = 1 \) and \( \mathbf{F}^p_{\text{en}}(\mathbf{X}, 0) = 1 \).

9. Specialization of the constitutive equations

In this section, based on experience with existing recent theories of viscoplasticity with isotropic and kinematic hardening, we specialize our constitutive theory by imposing additional constitutive assumptions.
9.1. Free-energy $\psi^{(e)}$

The spectral representation of $C^e$ is

$$C^e = \sum_{i=1}^{3} \omega_i^e r_i^e \otimes r_i^e, \quad \text{with} \quad \omega_i^e = \ln \lambda_i^e,$$

(9.1)

where $(r_1^e, r_2^e, r_3^e)$ are the orthonormal eigenvectors of $C^e$ and $U^e$, and $(\lambda_1^e, \lambda_2^e, \lambda_3^e)$ are the positive eigenvalues of $U^e$. Instead of using the invariants $x^e_C$, the free-energy $\psi^{(e)}$ for isotropic materials may be alternatively expressed in terms of the principal stretches as

$$\psi^{(e)} = \tilde{\psi}^{(e)}(\lambda_1^e, \lambda_2^e, \lambda_3^e).$$

(9.2)

Then, by the chain-rule and (8.2)$_2$, the stress $T^e$ is given by

$$T^e = 2 \frac{\partial \tilde{\psi}^{(e)}(\lambda_1^e, \lambda_2^e, \lambda_3^e)}{\partial C} = 2 \sum_{i=1}^{3} \frac{\partial \tilde{\psi}^{(e)}(\lambda_1^e, \lambda_2^e, \lambda_3^e)}{\partial \lambda_i^e} \frac{\partial \lambda_i^e}{\partial C} = \sum_{i=1}^{3} \frac{1}{\lambda_i^e} \frac{\partial \tilde{\psi}^{(e)}(\lambda_1^e, \lambda_2^e, \lambda_3^e)}{\partial \lambda_i^e} \frac{\partial C}{\partial C}.$$ \hspace{1cm} (9.3)

Assume that the squared principal stretches $\omega_i^e$ are distinct, so that the $\omega_i^e$ and the principal directions $r_i^e$ may be considered as functions of $C^e$; then

$$\frac{\partial \omega_i^e}{\partial C} = r_i^e \otimes r_i^e,$$

(9.4)

and, granted this, (9.4) and (9.3) imply that

$$T^e = \sum_{i=1}^{3} \frac{1}{\lambda_i^e} \frac{\partial \tilde{\psi}^{(e)}(\lambda_1^e, \lambda_2^e, \lambda_3^e)}{\partial \lambda_i^e} r_i^e \otimes r_i^e.$$ \hspace{1cm} (9.5)

Further, from (8.2)$_1$,

$$T = J^{-1} F^e T F^{eT} = J^{-1} R^u U^e T U^e R^{eT} = J^{-1} R^e \left( \sum_{i=1}^{3} \frac{1}{\lambda_i^e} \frac{\partial \tilde{\psi}^{(e)}(\lambda_1^e, \lambda_2^e, \lambda_3^e)}{\partial \lambda_i^e} r_i^e \otimes r_i^e \right) R^{eT}.$$ \hspace{1cm} (9.6)

Next, since $M^e = C^e T^e$ (cf., (8.3)), use of (9.1) and (9.5) gives the Mandel stress as

$$M^e = \sum_{i=1}^{3} \frac{\partial \tilde{\psi}^{(e)}(\lambda_1^e, \lambda_2^e, \lambda_3^e)}{\partial \lambda_i^e} r_i^e \otimes r_i^e.$$ \hspace{1cm} (9.7)

Let

$$E^e \overset{\text{def}}{=} \ln U^e = \sum_{i=1}^{3} E_i^e r_i^e \otimes r_i^e,$$

(9.8)

denote the logarithmic elastic strain with principal values

$$E_i^e \overset{\text{def}}{=} \ln \lambda_i^e,$$

(9.9)

and consider an elastic free-energy function of the form

$$\tilde{\psi}^{(e)}(\lambda_1^e, \lambda_2^e, \lambda_3^e) = \tilde{\psi}^{(e)}(E_1^e, E_2^e, E_3^e),$$

(9.10)

so that, using (9.7),

$$M^e = \sum_{i=1}^{3} \frac{\partial \tilde{\psi}^{(e)}(E_1^e, E_2^e, E_3^e)}{\partial E_i^e} r_i^e \otimes r_i^e.$$ \hspace{1cm} (9.11)
We consider the following simple generalization of the classical strain-energy function of infinitesimal isotropic elasticity which uses a logarithmic measure of finite strain\(^1\)
\[
\psi^{(p)}(E^p) = G|E_0^p|^2 + \frac{1}{2}K(\text{tr}E^p)^2,
\]
where
\[
G > 0 \quad \text{and} \quad K > 0,
\]
are the shear modulus and bulk modulus, respectively. Then, (9.11) gives
\[
M^p = 2GE_0^p + K(\text{tr}E^p)1
\]
and on account of (9.6), (9.7), and (9.14),
\[
T = J^{-1}R^pM^pR^pT.
\]

9.2. Free-energy \(\psi^{(p)}\)

The spectral representation of \(\mathbf{B}_e^p\) is
\[
\mathbf{B}_e^p = \sum_{i=1}^{3} b_i \mathbf{l}_i \otimes \mathbf{l}_i, \quad \text{with} \quad b_i = a_i^2,
\]
where \((\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3)\) are the orthonormal eigenvectors of \(\mathbf{B}_e^p\) and \(\mathbf{V}_e^p\), and \((a_1, a_2, a_3)\) are the positive eigenvalues \(\mathbf{V}_e^p\). Instead of using the invariants \(\mathbf{J}_e^p\), the free-energy \(\psi^{(p)}\) may alternatively be expressed as
\[
\psi^{(p)} = \psi^{(p)}(a_1, a_2, a_3).
\]
Then, by the chain-rule
\[
\frac{\partial \psi^{(p)}(a_1, a_2, a_3)}{\partial \mathbf{B}_e^p} = \frac{3}{2} \sum_{i=1}^{3} \frac{\partial \psi^{(p)}(a_1, a_2, a_3)}{\partial a_i} \frac{\partial a_i}{\partial \mathbf{B}_e^p} - \frac{1}{2} \sum_{i=1}^{3} \frac{\partial \psi^{(p)}(a_1, a_2, a_3)}{\partial a_i} \frac{\partial b_i}{\partial \mathbf{B}_e^p}.
\]
Assume that \(b_i\) are distinct, so that the \(b_i\) and the principal directions \(\mathbf{l}_i\) may be considered as functions of \(\mathbf{B}_e^p\). Then,
\[
\frac{\partial b_i}{\partial \mathbf{B}_e^p} = \mathbf{l}_i \otimes \mathbf{l}_i,
\]
and, granted this, (9.18) implies that
\[
\frac{\partial \psi^{(p)}(a_1, a_2, a_3)}{\partial \mathbf{B}_e^p} = \frac{1}{2} \sum_{i=1}^{3} \frac{1}{a_i} \frac{\partial \psi^{(p)}(a_1, a_2, a_3)}{\partial a_i} \mathbf{l}_i \otimes \mathbf{l}_i.
\]
Also, use of (9.16) and (9.20) in (8.4) gives the deviatoric back-stress as
\[
\mathbf{S}_e^p = \left( \sum_{i=1}^{3} a_i \frac{\partial \psi^{(p)}(a_1, a_2, a_3)}{\partial a_i} \mathbf{l}_i \otimes \mathbf{l}_i \right).
\]
Let
\[
\mathbf{E}_e^p \equiv \ln \mathbf{V}_e^p = \sum_{i=1}^{3} \ln a_i \mathbf{l}_i \otimes \mathbf{l}_i.
\]
\(^1\) Vladimirov et al. (2008) employ a Neo-Hookean free-energy function modified for elastic compressibility (cf., their Eq. (28)). Here, as an alternate, we use the classical strain-energy function for infinitesimal elasticity, but employ the large deformation Hencky-logarithmic elastic strain \(E^e\). Anand (1979, 1986) has shown that the simple free-energy function (9.12) is useful for moderately large elastic stretches. Additionally, use of the logarithmic strain helps in developing implicit time-integration procedures based on the exponential map (cf., Appendix A).
denote a logarithmic energetic strain with principal values \((\ln a_i)\). Note that since \(\text{tr}(\mathbf{E}^p_{en}) = a_1 a_2 a_3 = 1\),
\[
\text{tr}(\mathbf{E}^p_{en}) = \ln a_1 + \ln a_2 + \ln a_3 = \ln(a_1 a_2 a_3) = 0,
\]
and hence the strain tensor \(\mathbf{E}^p_{en}\) is \textit{traceless}. Next, consider the following simple defect energy:
\[
\psi^p(\mathbf{E}^p_{en}) = B \left[ (\ln a_1)^2 + (\ln a_2)^2 + (\ln a_3)^2 \right],
\]
with \(B > 0\), which, since \(\mathbf{E}^p_{en}\) is deviatoric, is analogous to (9.12). Then using (9.21), (9.22), and (9.23)
\[
\mathbf{S}_{en} = 2B\mathbf{E}^p_{en};
\]
we call the positive-valued constitutive parameter \(B\) the \textit{back-stress modulus}. Then the plastic Mandel stress (4.33) becomes
\[
\mathbf{M}^p_{en} = \mathbf{F}^p_{en} \mathbf{S}_{en} \mathbf{F}^p_{en}^{-1} = \mathbf{R}^p_{en} \left( \mathbf{V}^p_{en} (2B\mathbf{E}^p_{en}) \mathbf{V}^p_{en}^{-1} \right) \mathbf{R}^p_{en},
\]
or
\[
\mathbf{M}^p_{en} = \mathbf{R}^p_{en} \mathbf{S}_{en} \mathbf{R}^p_{en}.
\]

9.3. \textit{Strength functions}

First consider the strength relation
\[
|\langle \mathbf{M}^e_{eff} \rangle_0| = Y(d^p, S),
\]
appearing in (8.9). Define an \textit{equivalent shear stress} by
\[
\bar{\tau} \overset{\text{def}}{=} \frac{1}{\sqrt{2}} |\langle \mathbf{M}^e_{eff} \rangle_0|,
\]
and an \textit{equivalent shear strain rate} by
\[
\nu^p \overset{\text{def}}{=} \sqrt{2} d^p = \sqrt{2} |\mathbf{D}^p|,
\]
respectively. Using the definitions (9.28) and (9.29), we rewrite the strength relation (9.27) as
\[
\bar{\tau} = Y(\nu^p, S),
\]
and assume that the \textit{shear flow strength} \(Y(\nu^p, S)\) has the simple form
\[
Y(\nu^p, S) = g(\nu^p) S,
\]
where \(S\) now represents an isotropic flow resistance in shear, and
\[
g(0) = 0, \quad g(\nu^p) \text{ is a strictly increasing function of } \nu^p.
\]
We refer to the dimensionless function \(g(\nu^p)\) as the \textit{rate-sensitivity function}. Next, by (9.32) the function \(g(\nu^p)\) is \textit{invertible}, and the inverse function \(f = g^{-1}\) is strictly increasing and hence strictly positive for all nonzero arguments; further \(f(0) = 0\). Hence the relation (9.30) may be inverted to give an expression
\[
\nu^p = g^{-1} \left( \frac{\bar{\tau}}{\bar{S}} \right) \equiv f \left( \frac{\bar{\tau}}{\bar{S}} \right)
\]
for the equivalent plastic shear strain rate.

An example of a commonly used rate-sensitivity function is the \textit{power-law function}
\[
g(\nu^p) = \left( \frac{\nu^p}{\nu_0} \right)^m,
\]

\footnote{We absorb inconsequential factors of \(\sqrt{2}\) in writing (9.30).}
where \( m > 0 \), a constant, is a rate-sensitivity parameter and \( v_0 > 0 \), also a constant, is a reference flow-rate. The power-law function allows one to characterize nearly rate-independent behavior, for which \( m \) is very small. Further, granted the power-law function (9.34), the expression (9.33) has the specific form

\[
v^p = v_0 \left( \frac{\tau}{S} \right)^m .
\]  
(9.35)

Next consider the strength relation

\[
|M_{en}^p| = Y_{dis}(d^p_{dis}, d^p, S),
\]  
(9.36)

appearing in (8.12). Define another equivalent shear stress by

\[
\tau_{en} \overset{\text{def}}{=} \frac{1}{\sqrt{2}} |M_{en}^p|,
\]  
(9.37)

and another equivalent shear strain rate by

\[
v^p_{dis} = \sqrt{2} d^p_{dis} = \sqrt{2} |D_{dis}^p|,
\]  
(9.38)

respectively. Using the definitions (9.37) and (9.38), we rewrite the strength relation (9.36) as

\[
\tau_{en} = Y_{dis}(v^p_{dis}, v^p, S).
\]  
(9.39)

A specialization of \( Y_{dis}(v^p_{dis}, v^p, S) \) which leads to Armstrong and Frederick (1966) type kinematic hardening is

\[
Y_{dis}(v^p_{dis}, v^p, S) = \left( \frac{B}{\gamma v^p} \right) v^p_{dis},
\]  
(9.40)

where \( B \) is the back-stress modulus (cf., (9.25)), and \( \gamma \geq 0 \) is a dimensionless constant. Using (9.40), (9.39) becomes

\[
\tau_{en} = \left( \frac{B}{\gamma v^p} \right) v^p_{dis},
\]  
(9.41)

so that the term \( B/\gamma v^p \) represents a “pseudo”-viscosity (Dettmer and Reese, 2004). The relation (9.41) may be inverted to give

\[
v^p_{dis} = \left( \frac{\tau_{en}}{B} \right) \gamma v^p,
\]  
(9.42)

which shows that \( v^p_{dis} = 0 \) when either \( \gamma = 0 \) or when \( v^p = 0 \). Thus note that the constitutive assumption (9.42) constrains the theory such that \( D_{dis}^p = 0 \) whenever \( D^p = 0 \), and this in turn will lead to no change in the back-stress \( S_{en} \) when \( D^p = 0 \), that is when there is no macroscopic plastic flow.

9.4. Evolution equation for \( S \)

The evolution equation for the isotropic deformation resistance is specialized in a rate-independent form as

\[
\dot{S} = h(S)v^p
\]  
(9.43)

with \( h(S) \) a hardening function, which we take to be given by (Brown et al., 1989):

\[
h(S) = \begin{cases} 
    h_0(1 - \frac{\dot{S}}{S})^a & \text{for } S_0 \leq S \leq S^*, \\
    0 & \text{for } S \geq S^*,
\end{cases}
\]  
(9.44)

\footnote{Again absorbing inconsequential factors of \( \sqrt{2} \) in writing (9.39).}
where $S'$, $a$, and $h_0$ are constant moduli with $S' > S_0$, $a > 1$, and $h_0 > 0$. The hardening function (9.44) is strictly decreasing for $S_0 \leq S \leq S'$ and vanishes for $S \geq S'$.

9.5. Dissipation inequality

Finally, using (9.28), (9.29), (9.37), (9.38), and (9.42), the dissipation inequality (7.18) reduces to

$$
\left( \frac{\dot{\tau}}{B} + \frac{\gamma}{c} (\tau_{en})^2 \right) \psi > 0.
$$

(9.45)

9.6. Summary of the specialized constitutive model

In this section, we summarize the specialized form of our theory, which should be useful in applications. The theory relates the following basic fields:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{x}$</td>
<td>$\mathbf{z}(\mathbf{X}, t)$, motion;</td>
</tr>
<tr>
<td>$\mathbf{F}$</td>
<td>$\nabla \mathbf{z}$, deformation gradient;</td>
</tr>
<tr>
<td>$\mathbf{F}^e$, $f^e$</td>
<td>$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$; elastic–plastic decomposition of $\mathbf{F}$;</td>
</tr>
<tr>
<td>$\mathbf{F}^p$, $f^p$</td>
<td>$f^e = \det \mathbf{F}^e = J &gt; 0$, elastic distortion;</td>
</tr>
<tr>
<td>$\mathbf{F}^e$</td>
<td>$f^p = \det \mathbf{F}^p = 1$, inelastic distortion;</td>
</tr>
<tr>
<td>$\mathbf{F}^e$, $\mathbf{D}^e$</td>
<td>$\mathbf{F} = \mathbf{R}^e \mathbf{U}^e$, polar decomposition of $\mathbf{F}^e$;</td>
</tr>
<tr>
<td>$\mathbf{F}^p$, $\mathbf{D}^p$</td>
<td>$\mathbf{D}^p = \text{sym}(\mathbf{F}^p \mathbf{F}^{p-1})$, plastic stretching corresponding to $\mathbf{F}^p$;</td>
</tr>
<tr>
<td>$\mathbf{F}^e_{en}$, $\mathbf{F}^p_{en}$</td>
<td>$\mathbf{F}^e_{en} \mathbf{F}^p_{en} = 1$, elastic–plastic decomposition of $\mathbf{F}^p$;</td>
</tr>
<tr>
<td>$\mathbf{F}^p_{dis}$, $\mathbf{D}^p_{dis}$</td>
<td>$\mathbf{F}^p_{dis} \mathbf{D}^p_{dis} = 1$, energetic part of $\mathbf{F}^p$;</td>
</tr>
<tr>
<td>$\mathbf{T}$</td>
<td>$\mathbf{T} = \mathbf{T}^e$, Cauchy stress;</td>
</tr>
<tr>
<td>$\psi$, $S$</td>
<td>$\psi$, free-energy density per unit volume of intermediate structural space; scalar internal variable.</td>
</tr>
</tbody>
</table>

(1) Free energy.

We consider a separable free energy

$$
\psi = \psi^{(e)} + \psi^{(p)}.
$$

(9.46)

With

$$
\mathbf{U} = \sum_{i=1}^{3} \mathbf{z}_i \mathbf{r}_i \otimes \mathbf{r}_i,
$$

(9.47)

denoting the spectral representation of $\mathbf{U}$, and with

$$
\mathbf{E}^e = \sum_{i=1}^{3} \ln \mathbf{z}_i \mathbf{r}_i \otimes \mathbf{r}_i,
$$

(9.48)

denoting an elastic logarithmic strain measure, we adopt the following special form for the free-energy $\psi^{(e)}$:

$$
\psi^{(e)} = \frac{1}{2} G |\mathbf{E}_0^e|^2 + \frac{1}{2} K (\text{tr} \mathbf{E}^e)^2.
$$

(9.49)

where

$$
G > 0, \quad \text{and} \quad K > 0,
$$

(9.50)

are the shear modulus and bulk modulus, respectively. Further, with

$$
\mathbf{V}^p_{en} = \sum_{i=1}^{3} a_i \mathbf{l}_i \otimes \mathbf{l}_i,
$$

(9.51)
denoting the spectral representation of $\mathbf{V}_p$, and with

$$
E_p = \sum_{i=1}^{3} \ln a_i I_i \otimes I_i, \quad \text{tr} E_p = 0,
$$

we adopt a free-energy $\psi^{(p)}$ of the form

$$
\psi^{(p)} = B |E_p|^2,
$$

where the positive-valued parameter

$$
B > 0,
$$

is a back-stress modulus.


Corresponding to the special free-energy functions considered above, the Cauchy stress is given by

$$
T = J^{-1} \mathbf{R}^{ef} \mathbf{M}^e \mathbf{R}^{ef^T},
$$

where

$$
\mathbf{M}^e = 2G \mathbf{E}_0^e + K(\text{tr} \mathbf{E}^e) \mathbf{1},
$$

is an elastic Mandel stress.

The symmetric and deviatoric back-stress is defined by

$$
\mathbf{S}_p = 2B E_p,
$$

and the driving stress for $\mathbf{D}^p$ is the effective stress given by

$$
\langle \mathbf{M}^e_{\text{eff}} \rangle = \mathbf{M}^e_0 - \mathbf{S}_p.
$$

The corresponding equivalent shear stress is given by

$$
\tau = \frac{1}{\sqrt{2}} |\langle \mathbf{M}^e_{\text{eff}} \rangle|.
$$

The driving stress for $\mathbf{D}^p_{\text{dis}}$ is the stress measure

$$
\mathbf{M}^e_{\text{dis}} = \mathbf{R}^e_{\text{dis}} \mathbf{S}_p \mathbf{R}^e_{\text{dis}}.
$$

The corresponding equivalent shear stress is given by

$$
\tau_{\text{dis}} = \frac{1}{\sqrt{2}} |\mathbf{M}^e_{\text{dis}}|.
$$

(3) \textit{Flow rules.}

The evolution equation for $\mathbf{F}^p$ is

$$
\dot{\mathbf{F}}^p = \mathbf{D}^p \mathbf{F}^p, \quad \mathbf{F}^p(\mathbf{X},0) = \mathbf{1},
$$

$$
\mathbf{D}^p = \mathbf{v}^p \left( \frac{\mathbf{M}^e_{\text{eff}}}{2\tau} \right),
$$

$$
\mathbf{v}^p = \mathbf{v}_0 \left( \frac{1}{\gamma} \right)^{1/m},
$$

where $\mathbf{v}_0$ is a reference plastic strain rate with units of 1/time, and $m$ is a strain-rate-sensitivity parameter.

The evolution equation for $\mathbf{F}^p_{\text{dis}}$ is

$$
\dot{\mathbf{F}}^p_{\text{dis}} = \mathbf{D}^p_{\text{dis}} \mathbf{F}^p_{\text{dis}}, \quad \mathbf{F}^p_{\text{dis}}(\mathbf{X},0) = \mathbf{1},
$$

$$
\mathbf{D}^p_{\text{dis}} = \mathbf{v}^p_{\text{dis}} \left( \frac{\mathbf{M}^e_{\text{dis}}}{2\tau_{\text{dis}}} \right),
$$

$$
\mathbf{v}^p_{\text{dis}} = \gamma \left( \frac{\tau_{\text{dis}}}{\beta} \right) \mathbf{v}^p,
$$

where $\gamma \geq 0$ is a dimensionless constant.
Evolution equation for the isotropic-hardening variable $S$.

The internal variable $S$ is taken to obey the evolution equation
\[
\dot{S} = h(S) \varphi, \quad S(X, 0) = S_0,
\]

where $S_0 > S_0$, $a$, and $h_0$ are constant moduli with $S_0 > S_0$, $a \geq 1$, and $h_0 > 0$.

10. Some numerical solutions

In Appendix A, we develop a semi-implicit time-integration procedure, and obtain an (approximate) algorithmic tangent for the special constitutive theory summarized in the previous section. The time-integration procedure and the associated algorithmic tangent have been implemented in the commercial finite element program Abaqus/Standard (2008) by writing a user material subroutine (UMAT). In this section, we investigate some salient properties of the model and the accompanying time-integration procedure.

10.1. Simple tension/compression

First, in order to demonstrate that the constitutive theory captures the various isotropic and kinematic-hardening effects, we carried out a numerical simulation of a symmetric strain-cycle in simple tension and compression. The material parameters used in our numerical study are:

\[
\begin{align*}
G &= 80 \text{ GPa}, & K &= 175 \text{ GPa}, \\
\nu_0 &= 0.001 \text{ s}^{-1}, & m &= 0.02, \\
S_0 &= 100 \text{ MPa}, & h_0 &= 1250 \text{ MPa}, & a &= 1 \quad S^* = 250 \text{ MPa}, \\
B &= 40 \text{ GPa}, & \gamma &= 400;
\end{align*}
\]

the value of $m = 0.02$ is intended to represent a nearly rate-independent material.

The axial strain-cycle was imposed between true strain limits of $\epsilon = \pm 0.02$ at an absolute axial true strain rate of $|\dot{\epsilon}| = 0.01 \text{ s}^{-1}$; a fixed time step $\Delta t = 0.01 \text{ s}$ was used in the numerical simulations. The calculations were performed for: (i) no hardening, $h_0 = 0$ and $B = 0$; (ii) isotropic hardening only, $h_0 \neq 0$ and $B = 0$; (iii) kinematic hardening only, $h_0 = 0$ and $B \neq 0$; and finally (iv) combined kinematic and isotropic hardening, $h_0 \neq 0$ and $B \neq 0$. Plots of the axial stress $\sigma$ versus the axial strain $\epsilon$ are shown in Fig. 1(a). With no hardening, we obtain an elastic-perfectly-plastic response, as expected. When isotropic hardening only is allowed, we observe an increase in the flow stress, and the stress level at which plastic flow recommences on strain reversal is equal in magnitude to the stress level from which the reversal in strain was initiated. In the case of kinematic hardening only, we again observe an increase in the flow stress, but this time the magnitude of the stress level at which plastic flow recommences on strain reversal is substantially smaller than the stress level from which the reversal in strain direction was initiated; a clear manifestation of the Bauschinger effect. When isotropic hardening is combined with kinematic hardening, we see the continual evolution in the flow stress associated with combined isotropic and kinematic hardening during forward straining, and a clear Bauschinger effect due to the kinematic hardening upon strain reversal.

10.2. Simple shear

Next, we concentrate on numerical solutions related to homogeneous simple shear. With respect to a rectangular Cartesian coordinate system with origin $\mathbf{o}$ and orthonormal base vectors $\{\mathbf{e}_i| i = 1, 3\}$, a simple shearing motion is described by

\[
\mathbf{X} = \mathbf{X} + (\nu \Gamma) X_2 \mathbf{e}_1,
\]

with $\nu$ a shear strain rate, and $\Gamma = \nu t$ is the amount of shear. Throughout this section we use the material parameters summarized in (10.1).
The specific goals of our numerical solutions related to simple shear are as follows:

- To further demonstrate that the constitutive theory captures the major features of isotropic and kinematic hardening in the case of simple reversed shear.
- To investigate the convergence and stability properties of the time-integration scheme in simple reversed shear.
- To demonstrate that the constitutive equations do not suffer from oscillations in the stress response at large shear strains.
- To demonstrate that the constitutive equations and the time-integration procedure are indeed objective.

Fig. 1. (a) Comparison of axial stress $\sigma$ versus axial strain $\varepsilon$ for various types hardening in a symmetric strain-cycle simulation in simple tension and compression. (b) Comparison of shear stress $T_{12}$ versus shear strain $\Gamma$ for various types of hardening in reversed simple shear.
First, we wish to further demonstrate that the constitutive theory captures the various isotropic and kinematic-hardening effects. To this end, we consider simple shear, as described above, for a time of 5 s at a shear strain rate of $\dot{\gamma} = 0.01 \, \text{s}^{-1}$ for a total shear strain $\Gamma = \dot{\gamma} t = 0.05$, and then at a shear strain rate of $\dot{\gamma} = -0.01 \, \text{s}^{-1}$ for another 5 s to complete one reversal of strain, while using a fixed time step $\Delta t = 0.01 \, \text{s}$. As for simple tension/compression, the calculations were performed for: (i) no hardening, $h_0 = 0$ and $B = 0$; (ii) isotropic hardening only, $h_0 \neq 0$ and $B = 0$; (iii) kinematic hardening only, $h_0 = 0$ and $B \neq 0$; and finally (iv) combined kinematic and isotropic hardening, $h_0 \neq 0$ and $B \neq 0$, and the results are shown in Fig. 1(b). Again, we observe the important qualitative features of each case in simple reversed shear, i.e. elastic-perfectly-plastic response in the case of no hardening, a continual increase in the flow stress in cases involving isotropic hardening, and a clear Bauschinger effect in cases involving kinematic hardening.

Next, we investigated the stability and convergence properties of our time-integration procedure. Since our integration procedure is semi-implicit rather than fully-implicit, one might be concerned that there may be a restrictive time step constraint for stability. To ascertain any time step restrictions, we performed a convergence test. Reversed simple shear to a maximum shear strain of $\Gamma = 0.05$ at a shear strain rate of $|\dot{\gamma}| = 0.01 \, \text{s}^{-1}$ was considered while increasingly coarsening the time step. The specific time steps $\Delta t$ of $1 \times 10^{-3}, \ 1 \times 10^{-2}, \ 2.5 \times 10^{-2}, \ 1 \times 10^{-1}, \ 2.5 \times 10^{-1}, \ \text{and} \ \ 1 \, \text{s}$ were considered; these time steps translate to shear strain increments of $1 \times 10^{-5}, \ 1 \times 10^{-4}, \ 2.5 \times 10^{-4}, \ 1 \times 10^{-3}, \ 2.5 \times 10^{-3}, \ \text{and} \ \ 1 \times 10^{-2}$, respectively. Fig. 2 shows the stress–strain curves corresponding to these cases. For increasing time steps, no evidence of instability is observed, but of course the accuracy of the solution degrades, especially in the regions of elastic–plastic transitions. The order of convergence is demonstrated in Fig. 3; since no exact solution is available, the solution obtained using a time step of $\Delta t = 1 \times 10^{-3} \, \text{s}$ was treated as the “exact” solution and used to calculate the pointwise error for the cases corresponding to the larger time steps. The error $E$ was then obtained by taking the L-infinity norm of the pointwise error. As seen in Fig. 3, the time-integration procedure is first-order accurate. Thus, while no stability problems are encountered, the accuracy of the integration procedures deteriorates, especially when large time increments are taken during periods of rapidly changing elastic–plastic transitions.

![Fig. 2. Comparison of solutions for reversed simple shear for various time steps.](image)
We wish next to demonstrate that the constitutive equations do not produce shear-stress oscillations at large strains – a feature which has plagued numerous previous large deformation constitutive theories for kinematic hardening. To this end, we performed a monotonic simple shear simulation to a very large final shear strain of $\gamma = 10$, at a shear strain rate of $\dot{\gamma} = 0.01 \text{s}^{-1}$ using a fixed time step $\Delta t = 0.1 \text{s}$. The shear $T_{12}$ and normal components, $T_{11}$ and $T_{22}$, of the Cauchy stress $T$ are plotted in Fig. 4(a). Clearly there are no oscillations in the $T_{12}$ versus $\gamma$ response; the shear stress rises and saturates (because of the chosen value of the material parameters) at a shear strain of $\gamma \approx 1.5$. We have also carried out a similar calculation with all other material parameters the same as before, but by setting the dynamic recovery parameter $\gamma = 0$; this corresponds to the case of “linear” kinematic hardening. The variation of the shear and normal components of the Cauchy stress for this case are plotted in Fig. 4(b). Here, since there is no dynamic recovery for the kinematic hardening, the stress levels for this physically unrealistic case get quite large, but the $T_{12}$ versus $\gamma$ curve still does not show any oscillations; the peak in this curve is due to the use of a logarithmic energetic strain $\ln(V_{en})$ and not due to any oscillatory behavior in the evolution of the back-stress.

Finally, in order to demonstrate the frame-indifference of the constitutive equations and numerical algorithm, we have carried out a numerical calculation for simple shear with superposed rigid rotation, which is described by Weber et al. (1990):

$$\mathbf{x} = \mathbf{Q}(t)[\mathbf{X} + (\dot{\mathbf{r}})X_2\mathbf{e}_1],$$

(10.3)

with

$$\mathbf{Q}(t) = (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) \cos(\omega t) + (\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1) \sin(\omega t) + \mathbf{e}_3 \otimes \mathbf{e}_3,$$

(10.4)

a rotation about the $e_3$-axis. In performing our numerical calculation, we used the material parameters listed in (10.1), and the calculation was performed with $\omega = 0.1 \pi$ radians per second and $\dot{\gamma} = 0.01$ per second, for $t \in [0, 20]$, so that $\theta \equiv \omega t \in [0, 2\pi]$ and $\gamma \equiv \dot{\gamma} t \in [0, 0.2]$. The calculation was carried by using a fixed time step of $\Delta t = 0.1 \text{s}$, which corresponds to a shear strain increment of $\Delta \gamma = 0.001$ and a rotation increment of $\Delta \theta = 3.6^\circ$. Snapshots of the initial and deformed geometry at a few representative stages of the simple shear plus superposed rotation are shown in Fig. 5. Let the solution for the stress corresponding to the motion (10.3) be denoted by $T(t)$. The result of the shear component of the “unrotated” Cauchy stress $(\mathbf{Q}(t) T(t) \mathbf{Q}(t))$ versus the amount of shear $\gamma$, obtained from the numerical calculation is shown in Fig. 6. Also shown in this figure is $T_{12}$ versus $\gamma$ for the motion (10.3) with $\mathbf{Q}(t) = 1$. The two shear stress versus shear strain curves are indistinguishable from each other; the
excellent agreement between the two calculations verifies the objectivity of the constitutive equations and time-integration procedure.

10.3. Cyclic loading of a curved bar

Next, in order to exercise the algorithmic tangent in a case of inhomogeneous deformation, we consider the cyclic deformation of a curved bar of circular cross-section. A finite element mesh for the curved bar is shown in Fig. 7(a). The bar, which has a diameter of 50 mm, curves through a radius of 100 mm for 90°, and then extends straight for another 100 mm. The mesh consists of 2184 Abaqus-C3D8R elements. The nodes on the face denoted as “fixed”-face in Fig. 7(a) have prescribed null displacements, $u_1 = u_2 = u_3 = 0$, while the boundary conditions on the face denoted as the “moved”-face were specified as

![Fig. 4. Simple shear to large shear strain (a) with dynamic recovery $\gamma \neq 0$; and (b) without dynamic recovery, $\gamma = 0$. Note that there are no shear-stress oscillations in either case.](image-url)
\[ \Gamma = 0, \theta = 0^\circ \quad \Gamma = 0.025, \theta = 45^\circ \quad \Gamma = 0.05, \theta = 90^\circ \]

\[ \Gamma = 0.075, \theta = 135^\circ \quad \Gamma = 0.1, \theta = 180^\circ \quad \Gamma = 0.125, \theta = 225^\circ \]

\[ \Gamma = 0.15, \theta = 270^\circ \quad \Gamma = 0.175, \theta = 315^\circ \quad \Gamma = 0.2, \theta = 360^\circ \]

Fig. 5. Simple shear with superposed rigid rotation. The deformed geometry at each stage is shown with a solid line, while the dashed line represents the initial geometry.

The calculation was run to a final time of \( t_f = 30 \) s, so that the \( u_2 \)-displacement on the “moved”-face goes through three complete cycles. The material properties used in the calculation were the same as those in the previous calculations (10.1). The deformed mesh at the point of maximum deflection is also shown in Fig. 7(a).

In this (and subsequent) multi-element calculations, a variable time-stepping integration procedure was employed. As in the paper by Lush et al. (1989), the calculation was performed with a slightly modified version of the Abaqus/Standard static analysis procedure. The modification was introduced to enhance the automatic time-stepping procedure in Abaqus to control the accuracy of the constitutive time-integration. This was done by using as a control measure the maximum equivalent plastic shear strain increment

\[
\Delta \gamma_{\max}^p = (\Delta t)(\gamma_{n+1}^p)_{\max}
\]

occurring at any integration point in the model during an increment. The value of \( \Delta \gamma_{\max}^p \) was kept close to a specified nominal value \( \Delta \gamma^p \). From Fig. 2, we note that the accuracy of the solution begins to
degrade for shear strain increments of around $1 \times 10^{-3}$; accordingly, we chose $\Delta \gamma_s^p$ to be half this value, $0.5 \times 10^{-3}$. The automatic time-stepping algorithm operated to keep the ratio

$$R \equiv \frac{\Delta \gamma_s^{p \text{max}}}{\Delta \gamma_s^p}$$

close to 1.0 by adjusting the size of the time increments.\(^{14}\)

As the curved bar is cyclically displaced in the $(\mathbf{e}_2, \mathbf{e}_3)$-plane at the “moved-face”, the section of the bar near the “fixed”-face is subjected to reversed-torsion, while the part of the bar towards the “moved”-face is subjected to reversed-bending, resulting in states of shear, tension, and compression in various parts of the body. With increasing displacement magnitude at the “moved”-face, inhomogeneous plastic deformation initiates and spreads in various disparate regions of the curved bar. The integrated reaction force in the 2-direction on the “moved”-face is plotted against the prescribed $u_2$-displacement in Fig. 7(b). This resulting overall cyclic load–displacement curve is the result of the complex evolution of the internal variables controlling the combined isotropic and kinematic hardening at the various sections of the bar which are undergoing plastic deformation. The overall load versus displacement response is smooth, and one can clearly see the effects of both kinematic hardening in the prominent Bauschinger effect and isotropic hardening in the continual increase in the magnitude of the load with cyclic deformation.

This example shows that our semi-implicit integration procedure, when coupled with our heuristic automatic time-stepping algorithm and our approximate algorithmic tangent, is effective in

\(^{14}\) After an equilibrium solution for a time increment $\Delta t_n = t_{n+1} - t_n$ was found, the value of the ratio $R$ was checked to determine whether this solution would be accepted or not. If $R$ was greater than 1.25, then the solution was rejected, and a new time increment was used that was smaller by the factor $0.85/R$. If $R \leq 1.25$, then the solution was accepted, and the value of $R$ was used to determine the first trial size of the next time increment. The following algorithm was used:

$$\begin{align*}
\text{if } 0.8 < R \leq 1.25 & \text{ then } \Delta t_{n+1} = \Delta t_n/R, \\
\text{if } 0.5 < R \leq 0.8 & \text{ then } \Delta t_{n+1} = 1.25\Delta t_n, \\
\text{if } R \leq 0.5 & \text{ then } \Delta t_{n+1} = 1.50\Delta t_n.
\end{align*}$$

Note that the measure $\Delta \gamma_s^{p \text{max}}$ was allowed to exceed the user specified value $\Delta \gamma_s^p$ by up to 25%. This was done to avoid having to recalculate increments that came out just slightly above the specified nominal value, but were otherwise essentially acceptable.
obtaining an accurate, stable, and efficient solution in an inhomogeneous large deformation implicit finite element calculation.

10.4. Bending and spring-back of an aluminum sheet

As our final numerical example, we consider a simple plane-strain sheet forming operation in which an aluminum sheet is plastically bent about a mandrel of fixed radius and then unloaded. We use the recent experimental data of Cao et al. (2008) for AA6111-T4 sheets to estimate the material parameters in our model. The fit of the model to the experimental data is shown in Fig. 8, and the estimated material parameters are listed in (10.8):

\[
\begin{align*}
G &= 25.5 \text{ GPa}, & K &= 68 \text{ GPa}, \\
\nu_0 &= 0.001 \text{ s}^{-1}, & m &= 0.02, \\
S_0 &= 67.55 \text{ MPa}, & h_0 &= 1200 \text{ MPa}, & a &= 1, & S' &= 156.7 \text{ MPa}, \\
B &= 1.09 \text{ GPa}, & \gamma &= 61.08.
\end{align*}
\] (10.8)

Fig. 7. (a) Undeformed and deformed meshes for the cyclic loading of a curved bar. (b) Resulting tip load versus tip displacement.
The initial configuration for our plane-strain sheet forming simulation is shown in Fig. 9(a). We consider a 1 mm thick sheet with a length of 75 mm, which is to be bent to a radius of 50 mm between a pair of matched-rigid dies; accordingly, the radius of the top and bottom dies is prescribed to be 50.5 mm and 49.5 mm, respectively. Due to the symmetry of the problem, we consider only half of the geometry with suitable boundary conditions at the symmetry-plane. The sheet is modeled using a mesh consisting of 1880 Abaqus-CPE4H plane-strain elements with seven elements through the sheet thickness, and the dies are modeled using rigid surfaces. Contact between the sheet and the dies was modeled as frictionless. In the first step of the sheet forming simulation, the top die is moved downward to bend the sheet completely around the bottom die. Fig. 9(b) shows the geometry of the initial and fully bent geometries of the sheet. In the second step, the top die is moved back upwards, and the sheet is allowed to spring-back. The unloaded sprung-back geometry is also shown in Fig. 9(b).

It is widely believed that plasticity models which do not account for kinematic hardening tend to incorrectly predict the amount of spring-back in simulations of sheet-metal forming operations (cf., e.g., Zhao and Lee, 2001). Thus, our large deformation theory which not only includes both kinematic and isotropic hardening but also accounts for large rotations, should be useful in improved modeling of spring-back phenomena in sheet forming operations.

11. Concluding remarks

In this paper, we have formulated a large deformation constitutive theory for combined isotropic and kinematic hardening based on the dual decomposition $F = F^eF^p$ and $F^p = F^p_{en}F^p_{dis}$. We now show that a similar kinematic-hardening theory may also be constructed without using the decomposition $F^p = F^p_{en}F^p_{dis}$. Recall the relation (6.3) for the evolution of $B^p_{en}$,

$$B^p_{en} = D^p_{en}B^p_{en} + B^p_{en}D^p_{en},$$

and the kinematical equation (2.28)
The relation (11.2) gives

$$D_p = D_{en}^p + \text{sym} \left( F_{en}^p D_{dis}^p F_{en}^{-1} \right).$$

The relation (11.2) gives

$$D_{en}^p = D_p - \text{sym} \left( F_{en}^p D_{dis}^p F_{en}^{-1} \right),$$

$$= D_p - \left( \frac{\nu_{dis}^p}{2 \tau_{en}} \right) \text{sym} \left( F_{en}^p M_{en}^p F_{en}^{-1} \right) \quad \text{(using (9.63))},$$

$$= D_p - \left( \frac{\nu_{dis}^p}{2 \tau_{en}} \right) \text{sym} \left( F_{en}^p R_{en}^p S_{en} R_{en}^{-1}^p F_{en}^{-1} \right) \quad \text{(using (9.60))},$$

$$= D_p - \left( \frac{\nu_{dis}^p}{2 \tau_{en}} \right) \text{sym} \left( V_{en}^p S_{en} V_{en}^{-1} \right),$$

$$= D_p - \left( \frac{\nu_{dis}^p}{2 \tau_{en}} \right) \text{sym} \left( V_{en}^p (B \ln B_{en}^p) V_{en}^{-1} \right) \quad \text{(using (9.57) and } E_{en}^p \equiv (1/2) \ln B_{en}^p).$$
\[ D^p = \frac{B_{\text{dis}}^p}{2\tau_{\text{en}}} (\ln B_{\text{en}}^p), \]
\[ = D^p - \left( \frac{1}{2} \gamma_{\text{en}}^p \right) (\ln B_{\text{en}}^p) \quad \text{using (9.42)}. \] (11.4)

Using (11.4) in (11.1) gives
\[ B_{\text{en}}^p = \left( D^p - \left( \frac{1}{2} \gamma_{\text{en}}^p \right) (\ln B_{\text{en}}^p) \right) B_{\text{en}}^p + B_{\text{en}}^p \left( D^p - \left( \frac{1}{2} \gamma_{\text{en}}^p \right) (\ln B_{\text{en}}^p) \right), \]
or
\[ B_{\text{en}}^p = D^p B_{\text{en}}^p + B_{\text{en}}^p D^p - \gamma_{\text{en}}^p B_{\text{en}}^p (\ln B_{\text{en}}^p). \] (11.5)

Thus, if we set
\[ A \equiv B_{\text{en}}^p, \]
we obtain the evolution equation
\[ \dot{A} = D^p A + A D^p - \gamma_{\text{en}}^p A (\ln A). \] (11.6)

Therefore, instead of using the decomposition \( F^p = F_{\text{en}}^p F_{\text{dis}}^p \) and developing a theory which gives rise to a back-stress \( S_{\text{en}} \) because of a defect free-energy \( \tilde{\psi}^p(B_{\text{en}}^p) \), one may develop an alternate theory based on an internal variable \( A \), with a corresponding defect free-energy \( \tilde{\psi}^p(A) \), and an evolution equation for \( A \) obeying (11.6); indeed, this is the track taken recently by Anand et al. (2008). However, we do note that the development of an implicit time-integration procedure for a constitutive theory based on \( A \) is not as straightforward as the one developed here for a theory utilizing the \( F^p = F_{\text{en}}^p F_{\text{dis}}^p \) decomposition.

Finally, as is abundantly clear from the extensive literature on hardening models for metal plasticity (cf., e.g., Chaboche, 2008), the simple theory with combined isotropic and kinematic hardening developed in this paper is only foundational in nature, and there are numerous specialized enhancements/modifications to the theory that need to be incorporated in order to match actual experimental data for different metals; we leave such work to the future.

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**Appendix A**

In this Appendix we develop a semi-implicit time-integration procedure, and obtain an (approximate) algorithmic tangent for the theory formulated in this paper. The time-integration procedure and the associated tangent are intended for use in the context of “implicit” finite element procedures. We have implemented the procedure described below in the commercial finite element program Abaqus/Standard (2008) by writing a user material subroutine (UMAT), and it is using this finite element program that we have carried out the calculations whose results are reported in the main body of the paper.

**A.1. Time-integration procedure**

The constitutive time-integration problem is that given \( \{T_n, F_n, S_n, (S_{\text{en}})_n, (F_{\text{dis}})_n\} \), as well as \( F_n \) and \( F_{n+1} \), at time \( t_n \), we need to calculate...
\{T_{n+1}, F_{p, n+1}^p, S_{n+1}, (S_{en})_{n+1}, (F_{dis})_{n+1}\}

at time \(t_{n+1} = t_n + \Delta t\).

The evolution equation for \(\dot{F}^p = D^pF^p\) is integrated by means of an exponential map (Weber and Anand, 1990)

\[
F_{n+1}^p = \exp(\Delta tD_{n+1}^p)F_n^p,
\]

with \(D_{n+1}^p = \dot{D}_{n+1}^p(\{\text{Meff}_{n+1}\}_0, S_{n+1})\); (12.1)

the inverse of \(F_{n+1}^p\) is then

\[
F_{n+1}^{-1} = \exp(-\Delta tD_{n+1}^p).
\]

Next, using \(F^p = F^p_{\text{trial}}\) and (12.1), the elastic deformation gradient at the end of the step is given by

\[
F_{n+1} = F_{\text{trial}} \exp(-\Delta tD_{n+1}^p),
\]

(12.3)

where

\[
F_{\text{trial}} = F_{n+1}F_{n}^{-1}
\]

(12.4)

is a trial value of \(F^p\) at the end of the step. The elastic right Cauchy–Green tensor and its trial value at the end of the step are

\[
C_{n+1}^e = F_{n+1}^eF_{n+1}^e, \quad C_{\text{trial}}^e = F_{\text{trial}}^eF_{\text{trial}}^e.
\]

Thus, using (12.3),

\[
C_{n+1}^e = \exp(-\Delta tD_{n+1}^p)C_{\text{trial}}^e \exp(-\Delta tD_{n+1}^p).
\]

(12.6)

To proceed further we make our first approximation:

(A1) To first order in \(\Delta t\), we approximate

\[
\exp(-\Delta tD_{n+1}^p) \approx (1 - \Delta tD_{n+1}^p).
\]

Hence, (12.6) becomes

\[
C_{n+1}^e \approx (1 - \Delta tD_{n+1}^p)C_{\text{trial}}^e(1 - \Delta tD_{n+1}^p),
\]

\[
\approx C_{\text{trial}} - \Delta tD_{n+1}^pC_{\text{trial}} - \Delta tC_{\text{trial}}^eD_{n+1}^p,
\]

\[
\approx C_{\text{trial}}(1 - \Delta tC_{\text{trial}}^eD_{n+1}^p).
\]

(12.8)

Next, consider the term \((C_{\text{trial}}^{-1}D_{n+1}^p)C_{\text{trial}}^e\), with \(E_{\text{trial}}^G = \frac{1}{2}(C_{\text{trial}} - 1)\) denoting a Green strain corresponding to \(C_{\text{trial}}^e\); we have \(C_{\text{trial}} = 1 + 2E_{\text{trial}}^G\), and hence

\[
C_{\text{trial}}^{-1}D_{n+1}^pC_{\text{trial}} = (1 - 2E_{\text{trial}}^G)D_{n+1}^p(1 + 2E_{\text{trial}}^G),
\]

\[
\approx D_{n+1}^p + 2D_{n+1}^p E_{\text{trial}}^G - 2E_{\text{trial}}^G D_{n+1}^p - 4E_{\text{trial}}^G D_{n+1}^p E_{\text{trial}}^G.
\]

(12.9)

We now make our second approximation:

(A2) We assume that \(|E_{\text{trial}}^G| \ll 1\), so that (12.9) may be approximated as

\[
C_{\text{trial}}^{-1}D_{n+1}^pC_{\text{trial}} \approx D_{n+1}^p.
\]

(12.10)

Using (12.10), we find that (12.8) reduces to

\[
C_{n+1}^e \approx C_{\text{trial}}(1 - 2\Delta tD_{n+1}^p).
\]

(12.11)

Further, using the symmetry of \(C_{n+1}^e\) we note that the symmetric tensors \(C_{\text{trial}}\) and \((1 - 2\Delta tD_{n+1}^p)\) commute, and hence share the same principal directions. Thus taking the logarithm of (12.11) we obtain
\[ E_{n+1}^e = E_{\text{trial}}^e + \frac{1}{2} \ln (1 - 2\Delta t D_{n+1}^p), \] (12.12)

where

\[ E_{n+1}^e = \frac{1}{2} \ln C_{n+1}^e \quad \text{and} \quad E_{\text{trial}}^e = \frac{1}{2} \ln C_{\text{trial}}^e. \] (12.13)

Next, let

\[ C \overset{\text{def}}{=} 2G \left( \left[ \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right] + K \mathbf{1} \otimes \mathbf{1} \right) \] (12.15)

denote the elasticity tensor, and let

\[ M_{n+1}^e = C[E_{n+1}^e] \quad \text{and} \quad M_{\text{trial}}^e = C[E_{\text{trial}}^e], \] (12.16)

respectively, denote the elastic Mandel stress at the end of the time step, as well as its trial value. Then, operating on (12.14) by \( C \) gives

\[ M_{n+1}^e = M_{\text{trial}}^e - \Delta t C[D_{n+1}^p]. \] (12.17)

Subtracting \( (S_{en})_{n+1} \) from the left-hand side and \(^{15}\)

\[ (S_{en})_{\text{trial}} \equiv (S_{en})_n, \quad \text{(trial values are evaluated with plastic flow frozen)} \]

from the right-hand side of (12.17), and writing

\[ (M_{\text{eff}}^e)_{\text{trial}} \overset{\text{def}}{=} M_{\text{trial}}^e - (S_{en})_{\text{trial}}, \] (12.18)

gives

\[ (M_{\text{eff}}^e)_{n+1} = (M_{\text{eff}}^e)_{\text{trial}} - \Delta t C[D_{n+1}^p]. \] (12.19)

Next, from (9.62) we have

\[ D_{n+1}^p = \frac{1}{\sqrt{2}} v_{n+1}^p N_{n+1}^p, \quad N_{n+1}^p = \frac{((M_{\text{eff}}^e)_{n+1})_0}{\sqrt{2} \tau_{n+1}}, \quad \tau_{n+1} = \frac{1}{\sqrt{2}} |((M_{\text{eff}}^e)_{n+1})_0|, \] (12.20)

where

\[ v_{n+1}^p = \sqrt{2} |D_{n+1}^p|. \] (12.21)

Using (12.15) and (12.20) in (12.19) we obtain

\[ (M_{\text{eff}}^e)_{n+1} = (M_{\text{eff}}^e)_{\text{trial}} - \sqrt{2} G(\Delta t)^p N_{n+1}^p. \] (12.22)

Since \( N_{n+1}^p \) is deviatoric, the deviatoric and spherical parts of (12.22) give

\(^{15}\) Subtracting \( (S_{en})_n \) rather than \( (S_{en})_{n+1} \) from the right-hand side of (12.17) to arrive at (12.19), is a critical step in our time-integration procedure. It allows us to decouple the update of \( F^p \) and \( F_{\text{deu}}^p \) into two separate, yet otherwise implicit updates.
and (9.55) gives the update for the Cauchy stress as

\[
\left(\left(M_{\text{eff}}^e\right)_{n+1}\right)_0 = \left(\left(M_{\text{eff}}^e\right)_{\text{trial}}\right)_0 - \sqrt{2}G(\Delta t v_{\text{p}}^e)N_{n+1}^p,
\]

\[
\text{tr}(M_{\text{eff}}^e)_{n+1} = \text{tr}(M_{\text{eff}}^e)_{\text{trial}}.
\]

(12.23)

Using (12.20), (12.23) may be arranged as

\[
\sqrt{2}(\tau_{n+1} + G(\Delta t v_{\text{p}}^e))N_{n+1}^p = \left(\left(M_{\text{eff}}^e\right)_{\text{trial}}\right)_0.
\]

(12.24)

Next, defining

\[
N_{\text{trial}}^p = \frac{\left(\left(M_{\text{eff}}^e\right)_{\text{trial}}\right)_0}{\sqrt{2}\tau_{\text{trial}}}, \quad \tau_{\text{trial}} = \frac{1}{\sqrt{2}}(\text{tr}(M_{\text{eff}}^e)_{\text{trial}})|,
\]

(12.25)

(12.24) may be written as

\[
(\tau_{n+1} + G(\Delta t v_{\text{p}}^e))N_{n+1}^p = \tau_{\text{trial}}N_{\text{trial}}^p,
\]

which immediately gives

\[
N_{n+1}^p = N_{\text{trial}}^p,
\]

\[
\tau_{n+1} + G(\Delta t v_{\text{p}}^e) = \tau_{\text{trial}}.
\]

(12.27)

Thus the direction of plastic flow at the end of the step $N_{n+1}^p$ is determined by the trial direction of plastic flow $N_{\text{trial}}^p$.

Next, the implicit form of the flow strength relation (9.30) is

\[
\tau_{n+1} = Y(v_{n+1}^p, S_{n+1}).
\]

(12.28)

We integrate the evolution equation for $S$ in a semi-implicit fashion as

\[
S_{n+1} = S_n + h(S_n)(\Delta t v_{\text{p}}^e).
\]

(12.29)

Since the hardening function $h$ typically does not change rapidly, we do not expect the semi-implicit integration of the evolution equation for $S$ to have a significant effect on the stability of our algorithm.

Using (12.29) in (12.28), we have

\[
\tau_{n+1} = Y(v_{n+1}^p, (S_n + h(S_n)(\Delta t v_{\text{p}}^e))).
\]

(12.30)

Finally, using (12.30) in (12.27) gives the following implicit equation for $v_{n+1}^p$:

\[
\mathcal{G}(v_{n+1}^p) = \tau_{\text{trial}} - G(\Delta t v_{\text{p}}^e) - Y(v_{n+1}^p, (S_n + h(S_n)(\Delta t v_{\text{p}}^e))) = 0.
\]

(12.31)

Once a solution for $v_{n+1}^p$ is obtained by solving (12.31), $S_{n+1}$ is easily evaluated by using (12.29). The elastic Mandel stress $M_{n+1}^e$ is obtained from (12.17) using

\[
D_{n+1}^p = \frac{1}{\sqrt{2}}v_{n+1}^pN_{\text{trial}}^p,
\]

(12.32)

as

\[
M_{n+1}^e = M_{\text{trial}}^e - \sqrt{2}G(\Delta t v_{\text{p}}^e)N_{\text{trial}}^p.
\]

(12.33)

The update for $F^e$ is obtained from (12.1), using the result (12.32):

\[
F_{n+1}^e = \exp\left(\frac{1}{\sqrt{2}}(\Delta t v_{\text{p}}^e)N_{\text{trial}}^p\right)F_n^e.
\]

(12.34)

Next, calculating

\[
F_{n+1}^e = F_{n+1}F_{n+1}^{-1},
\]

(12.35)

and performing a polar decomposition of $F_{n+1}^e$ gives $R_{n+1}^e$. Finally, using $R_{n+1}^e$, and the relations (12.33) and (9.55) gives the update for the Cauchy stress as

\[
T_{n+1} = J_{n+1}^{-1}(R_{n+1}^e)(M_{n+1}^e)(R_{n+1}^e)^T \quad \text{with} \quad J_{n+1} = \text{det } F_{n+1}.
\]

(12.36)
We turn next to updating $S_{en}$ and $F^p_{\text{dis}}$. The evolution equation for $F^p_{\text{dis}} = D^p_{\text{dis}} F^p_{\text{dis}}$ is also integrated by using an exponential map and (9.63) as

\[ (F^p_{\text{dis}})_{n+1} = \exp \left( \Delta t \left( D^p_{\text{dis}} \right)_{n+1} \right) / (F^p_{\text{dis}})_n \] with \[ (D^p_{\text{dis}})_{n+1} = \left( \frac{\gamma}{2B} \right) v^p_{n+1} (M^p_{en})_{n+1} \];

(12.37)

the inverse of \( (F^p_{\text{dis}})_{n+1} \) is then

\[ (F^p_{\text{dis}})_{n+1}^{-1} = (F^p_{\text{dis}})_n^{-1} \exp \left( -\Delta t \left( D^p_{\text{dis}} \right)_{n+1} \right). \]

(12.38)

Hence, using $F^p_{en} = F^p_{\text{dis}}^{-1}$, the energetic part of $F^p$ at the end of the step is given by

\[ (F^p_{en})_{n+1} = (F^p_{en})_{\text{trial}} \exp \left( -\Delta t \left( D^p_{\text{dis}} \right)_{n+1} \right), \]

(12.39)

where

\[ (F^p_{en})_{\text{trial}} \overset{\text{def}}{=} F^p_{n+1} \left( F^p_{\text{dis}} \right)^{-1} \]

is a trial value of $F^p_{en}$ at the end of the step.

The tensors \( (F^p_{en})_{n+1} \) and \( (F^p_{en})_{\text{trial}} \) admit the polar decompositions

\[ (F^p_{en})_{n+1} = (R^p_{en})_{n+1} (U^p_{en})_{n+1}, \quad (F^p_{en})_{\text{trial}} = (R^p_{en})_{\text{trial}} (U^p_{en})_{\text{trial}}. \]

(12.41)

Using (12.41) in (12.39) and rearranging, we obtain

\[ (R^p_{en})_{n+1} (U^p_{en})_{n+1} \exp \left( \Delta t \left( D^p_{\text{dis}} \right)_{n+1} \right) = (R^p_{en})_{\text{trial}} (U^p_{en})_{\text{trial}}. \]

(12.42)

Next, from (9.57) and (9.60)

\[ M^p_{en} = R^T_{en} S_{en} R^p_{en} = R^T_{en} \left( 2B \ln U^p_{en} \right) R^p_{en} = 2B \ln U^p_{en}. \]

(12.43)

Thus the principal directions of \( (M^p_{en})_{n+1} \), and hence from \( (D^p_{\text{dis}})_{n+1} \) of \( (D^p_{\text{dis}})_{n+1} \), are the same as those \( (U^p_{en})_{n+1} \). Hence

\[ (U^p_{en})_{n+1} \exp \left( \Delta t \left( D^p_{\text{dis}} \right)_{n+1} \right) \]

is symmetric.

(12.44)

Then, because of the uniqueness of the polar decomposition theorem

\[ (R^p_{en})_{n+1} = (R^p_{en})_{\text{trial}}; \]
\[ (U^p_{en})_{n+1} \exp \left( \Delta t \left( D^p_{\text{dis}} \right)_{n+1} \right) = (U^p_{en})_{\text{trial}}. \]

(12.46)

Eq. (12.46) implies that \( (U^p_{en})_{n+1} \) and \( (U^p_{en})_{\text{trial}} \) have the same principal directions. Thus taking the logarithm on both sides and rearranging we have

\[ \ln (U^p_{en})_{n+1} = \ln (U^p_{en})_{\text{trial}} - \Delta t \left( D^p_{\text{dis}} \right)_{n+1}. \]

(12.47)

Multiplying (12.47) through by $2B$ and using (12.43) gives

\[ (M^p_{en})_{n+1} = (M^p_{en})_{\text{trial}} - 2B \Delta t \left( D^p_{\text{dis}} \right)_{n+1}, \quad \text{where} \quad (M^p_{en})_{\text{trial}} \overset{\text{def}}{=} 2B \ln (U^p_{en})_{\text{trial}}. \]

(12.48)

Finally, using (9.60), (12.37), \( (D^p_{\text{dis}})_{n+1} \), and pre-multiplying (12.48) by \( (R^p_{en})_{\text{trial}} \) and post-multiplying by \( (R^p_{en})_{\text{trial}} \) gives

\[ (S_{en})_{n+1} = (S_{en})_{n+1} - \gamma (\Delta t \gamma^p_{n+1}) (S_{en})_{n+1}, \quad \text{where} \quad (S_{en})_{\text{trial}} \overset{\text{def}}{=} 2B \ln (V^p_{en})_{\text{trial}}. \]

(12.49)

Thus, the back-stress $S_{en}$ is updated as

\[ (S_{en})_{n+1} = \left( \frac{1}{1 + \gamma (\Delta t \gamma^p_{n+1})} \right) (S_{en})_{\text{trial}}. \]

(12.50)

Correspondingly,

\[ (M^p_{en})_{n+1} = (R^p_{en})_{\text{trial}} (S_{en})_{n+1} (R^p_{en})_{\text{trial}}; \]

(12.51)

use of which in (12.37) provides the update \( (F^p_{\text{dis}})_{n+1} \) for $F^p_{\text{dis}}$. \]
Remark. Due to the use of the exponential map in integrating the evolution equations for $\mathbf{F}^p$ and $\mathbf{F}^p_{\text{dis}}$, the constraint of plastic incompressibility is exactly maintained by our time-integration procedure. This is easily verified by recalling the identity $\det(\exp \mathbf{A}) = \exp(\text{tr} \mathbf{A})$ and recognizing the deviatoric nature of $\mathbf{D}^p$ and $\mathbf{D}^p_{\text{dis}}$ in (12.1) and (12.37), respectively.

A.1.1. Summary of the time-integration procedure

- Given: $(\mathbf{T}_n, \mathbf{F}^p_n, \mathbf{S}_n, (\mathbf{F}^p_{\text{dis}})_{n-1})$, as well as $\mathbf{F}_n$ and $\mathbf{F}_{n+1}$, at time $t_n$.
- Calculate: $(\mathbf{T}_{n+1}, \mathbf{F}^p_n, \mathbf{S}_{n+1}, (\mathbf{F}^p_{\text{dis}})_{n+1})$ at time $t_{n+1} = t_n + \Delta t$.

Step 1. Calculate the trial elastic deformation gradient
$$\mathbf{F}^e_{\text{trial}} = \mathbf{F}_{n+1} \mathbf{F}^p_{n-1}. \quad (12.52)$$

Step 2. Perform the polar decomposition
$$\mathbf{F}^e_{\text{trial}} = \mathbf{R}^e_{\text{trial}} \mathbf{U}^e_{\text{trial}}. \quad (12.53)$$

Step 3. Calculate the trial elastic strain
$$\mathbf{E}^e_{\text{trial}} = \ln \mathbf{U}^e_{\text{trial}}. \quad (12.54)$$

Step 4. Calculate the trial Mandel stress and the trial effective Mandel stress
$$\mathbf{M}^e_{\text{trial}} = \mathbb{C} \mathbf{E}^e_{\text{trial}}, \quad (12.55)$$
$$\langle \mathbf{M}^e_{\text{eff}} \rangle_{\text{trial}} = \mathbf{M}^e_{\text{trial}} - (\mathbf{S}_n). \quad (12.56)$$

Step 5. Calculate the trial mean normal pressure, the deviatoric part of the trial effective Mandel stress, the trial equivalent shear stress, and the trial direction of plastic flow
$$\bar{p}_{\text{trial}} = -\frac{1}{2} \text{tr}(\langle \mathbf{M}^e_{\text{eff}} \rangle_{\text{trial}}), \quad (12.57)$$
$$
\langle \langle \mathbf{M}^e_{\text{eff}} \rangle_{\text{trial}} \rangle_0 = \langle \mathbf{M}^e_{\text{eff}} \rangle_{\text{trial}} + \bar{p}_{\text{trial}} \mathbf{1}, \quad (12.58)
$$
$$\tau_{\text{trial}} = \sqrt{\frac{1}{2} \langle \langle \mathbf{M}^e_{\text{eff}} \rangle_{\text{trial}} \rangle_0}, \quad (12.59)$$
$$\mathbf{N}^p_{\text{trial}} = \frac{\langle \langle \mathbf{M}^e_{\text{eff}} \rangle_{\text{trial}} \rangle_0}{\sqrt{2} \tau_{\text{trial}}}. \quad (12.60)$$

Step 6. Calculate $\mathbf{v}^p_{n+1}$ by solving
$$\mathcal{G}(\mathbf{v}^p_{n+1}) = \bar{\tau}_{\text{trial}} - G(\Delta t \mathbf{v}^p_{n+1}) - Y(\mathbf{v}^p_{n+1}, (\mathbf{S}_n + h(\mathbf{S}_n)(\Delta t \mathbf{v}^p_{n+1}))) = 0. \quad (12.61)$$

Step 7. Update $\mathbf{D}^p$
$$\mathbf{D}^p_{n+1} = \sqrt{(1/2) \mathbf{v}^p_{n+1} \mathbf{N}^p_{\text{trial}}}. \quad (12.62)$$

Step 8. Update $\mathbf{F}^p$
$$\mathbf{F}^p_{n+1} = \exp(\Delta t \mathbf{D}^p_{n+1}) \mathbf{F}^p_n. \quad (12.63)$$

Step 9. Update the Mandel stress $\mathbf{M}^e$
$$\mathbf{M}^e_{n+1} = \mathbf{M}^e_{\text{trial}} - \sqrt{2} G(\Delta t \mathbf{v}^p_{n+1}) \mathbf{N}^p_{\text{trial}}. \quad (12.64)$$

Step 10. Update $\mathbf{F}^e$
$$\mathbf{F}^e_{n+1} = \mathbf{F}^e_{n+1} \mathbf{F}^p_{n+1}. \quad (12.65)$$

Step 11. Perform the polar decomposition
$$\mathbf{F}^e_{n+1} = \mathbf{R}^e_{n+1} \mathbf{U}^e_{n+1}. \quad (12.66)$$
Step 12. Update the Cauchy stress

\[ f_{n+1} = \det(F_{n+1}^e), \]
\[ T_{n+1} = f_{n+1}^{-1}(R_{n+1}^e)(M_{n+1}^e)(R_{n+1}^e)^\top. \]  

(12.67)

(12.68)

Step 13. Update S

\[ S_{n+1} = S_n + h(S_n)(\Delta \mathbf{v}_n^p). \]  

(12.69)

Step 14. Calculate \((F_{en}^p)_{\text{trial}}\)

\[ (F_{en}^p)_{\text{trial}} = F_{n+1}^e (F_{dis}^p)^{-1}. \]  

(12.70)

Step 15. Perform the polar decomposition

\[ (F_{en}^p)_{\text{trial}} = (V_{en}^p)_{\text{trial}} (R_{en}^p)_{\text{trial}}. \]  

(12.71)

Step 16. Calculate the trial back-stress

\[ (S_{en})_{\text{trial}} = 2B \ln(V_{en}^p)_{\text{trial}}. \]  

(12.72)

Step 17. Update the back-stress \(S_{en}\)

\[ (S_{en})_{n+1} = \left( \frac{1}{1 + \gamma (\Delta \mathbf{v}_n^p)} \right) (S_{en})_{\text{trial}}. \]  

(12.73)

Step 18. Update the plastic Mandel stress \(M_{en}^p\)

\[ (M_{en}^p)_{n+1} = (R_{en}^p)_{\text{trial}} (S_{en})_{n+1} (R_{en}^p)_{\text{trial}}. \]  

(12.74)

Step 19. Update \(D_{dis}^p\)

\[ (D_{dis}^p)_{n+1} = \left( \frac{\gamma}{2B} \right) v_{n+1}^p (M_{en}^p)_{n+1}. \]  

(12.75)

Step 20. Update \(F_{dis}^p\)

\[ (F_{dis}^p)_{n+1} = \exp(\Delta \mathbf{r}(D_{dis}^p)_{n+1})(F_{dis}^p)_{n}. \]  

(12.76)

A.2. Jacobian matrix

In typical “implicit” finite element procedures utilizing a Newton-type iterative solution method, one needs to compute an algorithmically consistent tangent, often called the Jacobian matrix. We obtain an estimate for our Jacobian matrix below. From the outset we note that Jacobian matrices are used only in the search for the global finite element solution, and while an approximate Jacobian might affect the rate of convergence of the global iteration scheme, it will not impair the accuracy of our constitutive time-integration algorithm.

Consider the Cauchy stress at the end of the increment:

\[ T_{n+1} = (\det(F_{n+1}^e))^{-1} R_{n+1}^e M_{n+1}^e (R_{n+1}^e)^\top. \]  

(12.77)

Then with \(\Delta\) denoting a variation, (12.77) gives

\[ \Delta T_{n+1} = \left( (\Delta R_{n+1}^e)(R_{n+1}^e)^\top \right) T_{n+1} + T_{n+1} \left( (\Delta R_{n+1}^e)(R_{n+1}^e)^\top \right) + \text{tr} \left( (\Delta F_{n+1}^e) F_{n+1}^{e\top} \right) T_{n+1} \]

\[ = (\det(F_{n+1}^e))^{-1} R_{n+1}^e (\Delta M_{n+1}^e)(R_{n+1}^e)^\top. \]  

(12.78)

We assume that the variations \((\Delta R_{n+1}^e)(R_{n+1}^e)^\top\) and \(\text{tr} \left( (\Delta F_{n+1}^e) F_{n+1}^{e\top} \right)\) of the incremental elastic rotation and volume change are reasonably well-estimated by a commercial finite element program, such
as Abaqus/Standard (2008), in which we have implemented our constitutive model by writing a user material subroutine (UMAT). Thus, here we concentrate on evaluating the variation $\Delta M^e_{n+1}$ in (12.78), which is computed from

$$\Delta M^e_{n+1} = C [\Delta E^e_{\text{trial}}], \quad (12.79)$$

where the fourth-order tensor

$$C \overset{\text{def}}{=} \frac{\partial M^e_{n+1}}{\partial E^e_{\text{trial}}} \quad (12.80)$$

is the important constitutive contribution to the global Jacobian matrix.

Recall that the time-integration procedure gives

$$M^e_{n+1} = M^e_{\text{trial}} - \sqrt{2} G (\Delta t^p_{n+1}) N^p_{\text{trial}}. \quad (12.81)$$

Using (12.27), (12.81) may be written as

$$M^e_{n+1} = M^e_{\text{trial}} + \sqrt{2} (\tau_{n+1} - \tau_{\text{trial}}) N^p_{\text{trial}}. \quad (12.82)$$

By the product rule

$$C = \frac{\partial M^e_{\text{trial}}}{\partial E^e_{\text{trial}}} + \sqrt{2} (\tau_{n+1} - \tau_{\text{trial}}) \frac{\partial N^p_{\text{trial}}}{\partial E^e_{\text{trial}}} + N^p_{\text{trial}} \otimes \left( \sqrt{2} \left( \frac{\partial \tau_{n+1}}{\partial E^e_{\text{trial}}} - \frac{\partial \tau_{\text{trial}}}{\partial E^e_{\text{trial}}} \right) \right), \quad (12.83)$$

Since

$$M^e_{\text{trial}} = C [E^e_{\text{trial}}], \quad (12.84)$$

we have

$$\frac{\partial M^e_{\text{trial}}}{\partial E^e_{\text{trial}}} = C. \quad (12.85)$$

Next, since

$$N^p_{\text{trial}} = \frac{(M^e_{\text{eff}})^{\text{trial}}/0}{\sqrt{2} \tau_{\text{trial}}}, \quad (12.86)$$

we have

$$\frac{\partial N^p_{\text{trial}}}{\partial E^e_{\text{trial}}} = \sqrt{\frac{1}{2}} \left( \frac{1}{\tau_{\text{trial}}} \frac{\partial ((M^e_{\text{eff}})^{\text{trial}})/0}{\partial E^e_{\text{trial}}} - \frac{(M^e_{\text{eff}})^{\text{trial}}/0}{\tau_{\text{trial}}} \otimes \frac{\partial \tau_{\text{trial}}}{\partial E^e_{\text{trial}}} \right). \quad (12.87)$$

Now,

$$(M^e_{\text{eff}})^{\text{trial}} = M^e_{\text{trial}} - (S_{en})_n = C [E^e_{\text{trial}}] - (S_{en})_n \quad (12.88)$$

$$= \left( 2G \left( 1 - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right) + K \mathbf{1} \otimes \mathbf{1} \right) [E^e_{\text{trial}}] - (S_{en})_n.$$ 

Hence,

$$((M^e_{\text{eff}})^{\text{trial}}/0) = \left( 2G \left( 1 - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right) \right) [E^e_{\text{trial}}] - (S_{en})_n, \quad (12.89)$$

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16 Abaqus uses a hypoelastic constitutive equation for the stress of the type

$$\mathbf{T} = \mathbf{T} - \mathbf{W}^T \mathbf{T} \mathbf{W} = \mathbf{T} - \mathbf{W}^T \mathbf{T} \mathbf{W} = \mathbf{C} \cdot \mathbf{D} - \mathbf{D}^p.$$
and therefore
\[ \frac{\partial ((M_{eff}^t)_{trial})_0}{\partial E_{trial}} = 2G \left( \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right). \] (12.90)

Further,
\[ \bar{\tau}_{\text{trial}} = \sqrt{\frac{1}{2}} \left((M_{eff}^t)_{trial})_0 \right). \] (12.91)

Therefore, by the chain-rule
\[ \frac{\partial \bar{\tau}_{\text{trial}}}{\partial E_{trial}} = \sqrt{\frac{1}{2}} \left( \frac{\partial ((M_{eff}^t)_{trial})_0}{\partial E_{trial}} \right) \left( \frac{\partial ((M_{eff}^t)_{trial})_0}{\partial ((M_{eff}^t)_{trial})_0} \right) \frac{\partial ((M_{eff}^t)_{trial})_0}{\partial (M_{eff}^t)_{trial}}. \] (12.92)

Further,
\[ \frac{\partial \bar{\tau}_{\text{trial}}}{\partial E_{trial}} = \sqrt{\frac{1}{2}} 2G \left( \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right) \left( \frac{(M_{eff}^t)_{trial}}{(M_{eff}^t)_{trial}} \right) \frac{(M_{eff}^t)_{trial}}{(M_{eff}^t)_{trial}} \right), \] (12.93)

and therefore
\[ \frac{\partial \bar{\tau}_{\text{trial}}}{\partial E_{trial}} = \sqrt{\frac{1}{2}} 2G \left( \frac{(M_{eff}^t)_{trial}}{(M_{eff}^t)_{trial}} \right) \frac{(M_{eff}^t)_{trial}}{(M_{eff}^t)_{trial}} \right). \] (12.94)

Thus, using (12.90) and (12.96) in (12.87) we obtain
\[ \frac{\partial N_{p\text{trial}}}{\partial E_{\text{trial}}} = 2G \sqrt{2} \bar{\tau}_{\text{trial}} \left( \left( \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right) - N_{p\text{trial}} \otimes N_{p\text{trial}} \right), \] (12.97)

and substituting (12.85) and (12.97) in (12.83), we have
\[ C = \bar{C} + \left( \frac{\bar{\tau}_{n+1}}{\tau_{\text{trial}}} - 1 \right) \left( \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right) - N_{p\text{trial}} \otimes N_{p\text{trial}} \right) \]
\[ + N_{p\text{trial}} \otimes \left( \sqrt{2} \frac{(\partial \bar{\tau}_{n+1})}{(\partial E_{\text{trial}})} - \sqrt{2}Gn_{p\text{trial}} \right). \] (12.98)

Noting that \( E_{\text{trial}} \) enters the equations through \( \tau_{\text{trial}} \), we have, using (12.96)
\[ \frac{\partial \tau_{n+1}}{\partial E_{\text{trial}}} = \frac{\partial \tau_{n+1}}{\partial \tau_{\text{trial}}} \frac{\partial \tau_{\text{trial}}}{(\partial E_{\text{trial}})} = \frac{\partial \tau_{n+1}}{\partial \tau_{\text{trial}}} \sqrt{2}Gn_{p\text{trial}}. \] (12.99)

Finally, substituting (12.99) into (12.98), using (12.15), and rearranging, we have
\[ C = \bar{C} - 2G \left( \frac{\bar{\tau}_{n+1}}{\tau_{\text{trial}}} - \frac{\partial \tau_{n+1}}{\partial \tau_{\text{trial}}} \right) N_{p\text{trial}} \otimes N_{p\text{trial}} \] (12.100)

where
\[ \bar{C} = 2G \left( \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right) + K \mathbf{1} \otimes \mathbf{1} \text{ with } \frac{\bar{\tau}_{n+1}}{\tau_{\text{trial}}} \text{.} \] (12.101)

Thus, it remains to determine the derivative
\[ \frac{\partial \tau_{n+1}}{\partial \tau_{\text{trial}}} \] in (12.100). First, we rewrite the updates for \( \tau_{n+1}, (12.27) \), and \( S_{n+1}, (12.69) \), as
\[ \tau_{n+1} = \tau_{\text{trial}} - G\Delta t f \left( \frac{\tau_{n+1}}{S_{n+1}} \right), \]
\[ S_{n+1} = S_{n} + \Delta t h (S) f \left( \frac{\tau_{n+1}}{S_{n+1}} \right), \] (12.102)
respectively, where \( f \) is the flow function of (9.33). Using the definitions

\[
X \equiv \{ \tau_{n+1} \} \quad \text{and} \quad Y \equiv \{ \tau_{\text{trial}} \},
\]

(12.103)

the system of equations (12.102) may be written as

\[
X = G(X, Y),
\]

(12.104)

where

\[
G = \begin{cases} 
\tau_{\text{trial}} - G \Delta t f \left( \frac{\tau_{n+1}}{S_{n+1}} \right) 
\approx \Delta t h(S_n) f \left( \frac{\tau_{n+1}}{S_{n+1}} \right) 
\end{cases}
\]

(12.105)

Differentiating (12.104) with respect to \( Y \) (at the solution point) we obtain

\[
\frac{\partial X}{\partial Y} = \frac{\partial G(X, Y)}{\partial Y} + \frac{\partial G(X, Y)}{\partial X} \frac{\partial X}{\partial Y},
\]

(12.106)

from which

\[
\frac{\partial X}{\partial Y} = A^* \left( \frac{\partial G(X, Y)}{\partial Y} \right),
\]

(12.107)

where

\[
A^* = \left( I - \frac{\partial G(X, Y)}{\partial Y} \right)^{-1}.
\]

(12.108)

Straightforward calculations show

\[
\frac{\partial G(X, Y)}{\partial X} = \begin{bmatrix} -G \Delta t f \left( \frac{\tau_{n+1}}{S_{n+1}} \right) & -G \Delta t h(S_n) \left( \frac{\tau_{n+1}}{S_{n+1}} \right) \\ \Delta t h(S_n) f \left( \frac{\tau_{n+1}}{S_{n+1}} \right) & \Delta t h(S_n) f \left( \frac{\tau_{n+1}}{S_{n+1}} \right) \end{bmatrix}, \quad \text{and} \quad \frac{\partial G(X, Y)}{\partial Y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

(12.109)

Thus, we obtain that

\[
\frac{\partial \tau_{n+1}}{\partial \tau_{\text{trial}}} = A^*_{11},
\]

(12.110)

substitution of which in (12.100) completes our estimate for the Jacobian matrix \( \mathbf{C} \).

References

Abaqus/Standard, 2008. SIMULIA, Providence, RI.

17 Here \( I \) is the second-order identity matrix.


