Continuum thermomechanics of the nonlocal granular rheology

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Abstract

We formulate a nonlocal, or scale-dependent, elasto-viscoplastic continuum model for granular materials, consistent with the principles of modern continuum thermomechanics. Importantly, the theory contains a scalar, energetic order parameter, referred to as the granular fluidity. We assume power to be expended over the rate of change of the fluidity and its gradient and undertake a derivation based upon the principle of virtual power in the style of Gurtin (1996). This approach results in a non-standard microforce balance, which when combined with our choice of specific constitutive equations, takes the form of a partial differential relation that the fluidity must obey. Finally, we simplify the equations into a form appropriate for steady granular flows. The resulting boundary-value problem was previously shown to be capable of describing a wide array of experimental granular flow data.

1. Introduction

Granular materials are ubiquitous in engineering, integral in the geotechnical, food processing, pharmaceutical, and energy industries. These materials display a rich phenomenology in their mechanical response, with behavior ranging from solid-like – as grains at rest in a hopper – to fluid-like – as grains flowing down a chute. Moreover, aspects of the collective mechanical behavior of a granular media are crucially dependent upon the size of the constituent grains. These factors make formulating a continuum model for granular mechanics a particularly challenging undertaking. For a recent review of continuum models of granular deformation, see Goddard (2013).

One continuum approach to modeling the behavior of granular materials, developed by solid mechanicians, is that of soil mechanics (Schoefield and Wroth, 1968; Nedderman, 1992; Spencer, 1964; Rudnicki and Rice, 1975; Anand and Gu, 2000; Zhu et al., 2006). The plasticity theories of soil mechanics – aimed at describing the solid-like, slow deformation regime – typically are rate-independent, utilize a pressure-sensitive yield condition, and include plastic dilatancy. These models have been successfully utilized in describing the load-carrying capacity of granular materials and the orientation of shear bands. Another broad approach, which considers granular media from a more fluids-oriented viewpoint, is that of granular rheology (Bagnold, 1954; Jaeger et al., 1996; Halsey and Mehta, 2002), which describes the observable dependence of stress on strain-rate in dense granular flows. This type of approach is capable of describing the velocity fields of rapid flows, such as the flow of grains down a chute (Jop et al., 2006; Kamrin, 2010). However, granular media are amorphous materials, and as such, there is a smooth transition between the clearly solid-like behavior of the soil mechanics regime and the clearly fluid-like behavior of the rapid-flow regime. In this transition, slow, dense granular deformation can display elements of both regimes.
Further, mechanical behavior in this regime is markedly grain-size dependent (MiDi, 2004; Dijksman and van Hecke, 2010), and since both classes of approaches are scale-independent, they are incapable of describing these varied behaviors.

Grain-size dependence in the dense flow regime manifests itself in a number of interesting ways. Several examples are highlighted below:

1. The length-scales in the velocity fields in a wide variety of geometries – such as annular Couette cells (Mueth et al., 2000; MiDi, 2004), heap flows (Komatsu et al., 2001), plate-dragging experiments (Sivashski et al., 2006), and split-bottom cells (Fenistein and van Hecke, 2003; Fenistein et al., 2004; Fenistein et al., 2006; Cheng et al., 2006) – are strongly grain-size dependent, with shear bands determined by the grain size.

2. Flows of thin layers of grains down an inclined surface exhibit a size effect whereby thinner layers require more tilt to begin flowing (Pouliquen, 1999; MiDi, 2004; Silbert et al., 2003).

3. Drainage flows – such as those in out of a hopper, silo, or hourglass – display an important dependence of outflow rate on grain size (Beverloo et al., 1961; Choi et al., 2005).

4. Finally, a unique problem in granular flow is that of mechanically-induced creep (Nichol et al., 2010; Reddy et al., 2011). Granular materials display a distinct yield point, and therefore, one may apply a force below some critical value to an intruder, such as a cylindrical or spherical probe, without inducing flow. However, when one drives shear deformation in one region of a granular medium, this fluidizes quiescent regions far from the sheared zone enabling creep deformation when a force is applied to an intruder in the quiescent region – even if this force is far below the critical value.

Many models have been developed attempting to describe scale-dependent phenomenology in granular materials. Examples include nonlocal order-parameter models (Aranson and Tsimring, 2002; Kamrin and Bazant, 2007; Bouzid et al., 2013), extensions of kinetic theory (Savage, 1998; Jenkins and Berzi, 2010), Cosserat-plasticity-based models (Mohan et al., 2002), and gradient plasticity theories (Vardoulakis and Aifantis, 1991; Al Hattamleh et al., 2004; Hashiguchi and Tsutsumi, 2007). While some have had success in individual geometries, the ability to describe experimental observations across a range of geometries has been elusive. Recently, we proposed a continuum-level model for dense granular flows that is uniquely capable of predicting velocity fields in a wide variety of two-dimensional (Kamrin and Koval, 2012) and three-dimensional (Henann and Kamrin, 2013) flow configurations. The ingredients of our modeling approach are summarized below:

1. The theory is based upon the concept of the “granular fluidity” $g$ which we take to be a non-negative state variable with units of (1/time) that characterizes the susceptibility of a point in a granular media to flow. The granular fluidity may be thought of as behaving like an order parameter in a Ginzburg–Landau approach. The important choice of the granular fluidity as the appropriate parameter is based on successes in the emulsions community in using the concept of size-dependent fluidity (inverse viscosity) to model size-effects (Goyon et al., 2008; Bocquet et al., 2009). As with other order parameter approaches (Aranson and Tsimring, 2002; Bouzid et al., 2013), it bears noting that one may non-dimensionalize the granular fluidity $g$ by factoring out a constant local rearrangement time-scale, consistent with the original analysis for emulsions (Bocquet et al., 2009). We will adopt this approach in Section 2.4.

2. With the Cauchy stress denoted as $\boldsymbol{T}$, we define the stress ratio $\mu = \tau/P$, where $\tau = (T_0 : T_0/2)^{1/2}$ is the equivalent shear stress, $T_0$ the stress deviator, and $P = -(1/3)\text{tr}\boldsymbol{T}$ the pressure. As is standard for granular materials, we limit attention to stress states with positive pressure, $P > 0$. The granular fluidity plays the role of an inverse viscosity scaled by the pressure, relating the equivalent shear strain rate $\dot{\gamma}$ to the stress ratio $\mu$ as follows:

$$\dot{\gamma} = \frac{\mu}{g^2}.$$

To put this in a three-dimensional context, the symmetric strain-rate tensor $\boldsymbol{D}$, which is taken to be deviatoric ($\text{tr}\boldsymbol{D} = 0$) to model constant-volume, steady flow, is then given by

$$\boldsymbol{D} = \frac{\dot{\gamma}}{2\tau} \frac{T_0}{\text{tr}\boldsymbol{T}},$$

where we have assumed that the strain-rate and deviatoric Cauchy stress tensors are approximately codirectional (Jop et al., 2006; Rycroft et al., 2009; Kamrin, 2010; Henann and Kamrin, 2013). Using (1.1) and (1.2), we have that the Cauchy stress $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = -P\mathbf{1} + 2\left(\frac{P}{g}\right)\boldsymbol{D},$$

in which we see the pressure $P$ playing the role of a Lagrange multiplier enforcing the constant-volume constraint.

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1. The assumption that flow is a constant-volume process stems from our interest in modeling steady flow. While volume change is a well-known part of granular flow phenomenology, it is a transient effect, and discrete element simulations (Koval et al., 2009; Rycroft et al., 2009) have shown that well-developed, or steady, flows progress at constant volume. As a result, the constant-volume assumption is common in granular flow modeling (Aranson and Tsimring, 2002; Mohan et al., 2002; Kamrin and Bazant, 2007; Jenkins and Berzi, 2010; Kamrin, 2010; Henann and Kamrin, 2013).

2. Slight deviations from codirectionality have been observed in discrete element simulations, namely ~5% non-coaxiality between $\boldsymbol{D}$ and $T_0$ (Luding, 2007) and the normal stress differences (Depken et al., 2007).
3. In the case of homogeneous simple shearing, the rheology of dense granular flow has progressed to a significant understanding with which the model must agree, attributable as far back as Bagnold (1954) with renewed study in the last 10 years (MiDi, 2004). Basic dimensional arguments produce a one-to-one relationship between the stress ratio $\mu$ and the dimensionless strain rate $I = \gamma \sqrt{\frac{d^2 \rho_s}{P}}$, referred to as the inertial number with $d$ the mean grain diameter and $\rho_s$ the grain density. This yields a functional dependence $\mu = \mu(I)$, which when a Bingham-like relation is adopted (da Cruz et al., 2005), may be inverted as

$$\dot{\gamma} = \dot{\gamma}_{loc}(P, \mu) = \begin{cases} \sqrt{P/\rho_s} d^2 (\mu - \mu_s)/b & \text{if } \mu > \mu_s, \\ 0 & \text{if } \mu \leq \mu_s, \end{cases}$$

(1.4)

with dimensionless constant $b$ and $\mu_s$, a static yield value. This local rheology — referred to in this manner because it relates the local stress state to the local state of strain rate — may be used to define a local fluidity $g_{loc} = \dot{\gamma}_{loc}/\mu$. 

4. Significant deviation from this one-to-one correspondence is observed in inhomogeneous flow (Koval et al., 2009; Kamrin and Koval, 2012), motivating a nonlocal relation. Consequently, the granular fluidity is taken to obey the following differential relation

$$\nabla^2 g = \frac{1}{\xi} (g - g_{loc}).$$

(1.5)

Here $\nabla^2 (\cdot)$ denotes the Laplacian operator and $\xi(\mu)$ is a stress-dependent length-scale called the cooperativity length. The presence of the $\nabla^2 g$ term gives rise to a nonlocal effect, which maybe attributed to far-field motion causing disturbances in the forces emitted through the contact network (Bocquet et al., 2009). In the case of homogeneous flow, the variable $g$ collapses to a stress-dependent formula $g_{loc}$, which produces the local rheology that is now well-established from both a physical and dimensional scaling standpoint. When flow inhomogeneity is present, this differential relation captures the loss of local constitutive uniqueness in the relation between $\mu$ and $I$.

Our model in this form successfully rationalized an unprecedented amount of experimental granular flow data (Henann and Kamrin, 2013). However, the understanding of the continuum underpinning of the model remains unresolved. A major tenet in modern continuum mechanics is that equations should be clearly classified as balance laws or constitutive equations (Gurtin et al., 2010), and in its current form, the nature of the differential relation (1.5) is not clear. Indeed, like many scientific theories, the concept of nonlocal fluidity has revealed itself to be an effective modeling tool before a sufficiently rigorous derivation has been formulated.

The purpose of this paper is to provide a formulation of the nonlocal granular rheology based upon modern, rational continuum thermomechanics in an elasto-viscoplastic context. Adopting a dimensionless form of the granular fluidity $g$ as an order parameter, the theory may be considered a cousin of gradient plasticity (Gurtin and Anand, 2005b,c; Lele and Anand, 2009; Anand et al., 2012), based on gradients of the Ginzberg–Landau-type order parameter, rather than the plastic strain. Accordingly, we allow power to be expended over the rate of change of the dimensionless granular fluidity as well as the rate of change of its gradient. Following the work of Gurtin (1996), we then apply the principle of virtual power, which yields a microforce balance, which, after appropriate choices of constitutive equations and further simplification, leads to both our differential relation for the fluidity (1.5) and the local rheology (1.4). The paper is organized as follows. In Section 2, we outline the continuum framework, including the principle of virtual power arguments that lead to microforce balances as well as the derivation of a free-energy imbalance based on the second law. In Section 3, guided by the free-energy imbalance, we consider constitutive equations, which we make distinct from the balance laws of the previous section. Finally, in Section 4, we choose specific constitutive functions, which justify the form of the nonlocal rheology used in previous work (Kamrin and Koval, 2012; Henann and Kamrin, 2013).

2. Continuum framework

2.1. Basic kinematics

We consider a homogeneous body $B$ identified with the region of space it occupies in a fixed reference configuration and denote by $X$ an arbitrary material point of $B$. A motion of $B$ is then a smooth one-to-one mapping $x = \chi(X, t)$ to the deformed body $\mathcal{B}$, with deformation gradient given by $F = \nabla \chi$, such that $J = \det F > 0$. We base our theory on the Kröner (1960)–Lee (1969) decomposition of the deformation gradient $F$ into elastic and plastic parts,

$$F = FE^P,$$

(2.1)

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3 Our notational conventions follow those of Gurtin et al. (2010). The symbols $\nabla$ and $\text{Div}$ denote the gradient and divergence with respect to the material point $X$ in the reference configuration; grad and div denote these operators with respect to the point $x = \chi(X, t)$ in the deformed configuration. A superposed dot denotes the material time-derivative.
so that $J^e J^p$ where $J^e = \det F^e$ and $J^p = \det F^p$. The plastic distortion $F^p(X)$ represents a local deformation of the material element at $X$ through plastic mechanisms from the reference space to the intermediate, or structural, space. The material element is subsequently acted upon by the elastic distortion $F^e(X)$, representing elastic stretching and rotation from the intermediate to the current configuration.

The velocity and spatial velocity gradient are given by

$$v = \dot{X} \quad \text{and} \quad L = \text{grad} v = FF^{-1}, \quad (2.2)$$

so that by (2.1) and (2.2)

$$L = L^e + F^e L^p F^e, \quad (2.3)$$

with

$$L^e = \dot{F}^e F^e^{-1} \quad \text{and} \quad L^p = \dot{F}^p F^p^{-1}. \quad (2.4)$$

We denote the elastic and plastic stretching and spin tensors by

$$D^e = \text{sym} L^e, \quad W^e = \text{skw} L^e, \quad D^p = \text{sym} L^p, \quad W^p = \text{skw} L^p. \quad (2.5)$$

The polar decomposition of $F^e$ is

$$F^e = R^e U^e, \quad (2.6)$$

where $R^e$ is a rotation (a proper, orthogonal tensor) and $U^e$ is the symmetric, positive-definite right stretch tensor. Also, the right elastic Cauchy–Green tensor is given by

$$C^e = U^e^2 = F^e T F^e, \quad (2.7)$$

and the elastic Green strain is

$$E^e = \frac{1}{2} \left( C^e - 1 \right). \quad (2.8)$$

2.2. Small elastic strains

With an eye towards modeling flowing granular materials, we make the key kinematical restriction of small elastic strains, stated precisely in terms of the Green strain as

$$|E^e| \ll 1. \quad (2.9)$$

Under this assumption, all definitions of the elastic strain will reduce to the same quantity, which we will continue to refer to as $E^e$. While the elastic strains are small, the elastic rotation is not required to be identity, and therefore, under this approximation, the elastic distortion $F^e$ is dominated by the rotation, so that

$$F^e \approx R^e, \quad U^e \approx 1, \quad \text{and} \quad J^e \approx 1. \quad (2.10)$$

Under this approximation, (2.4), (2.6), and (2.8) lead to

$$L^e \approx R^e E^e R^e T + R^e R^e T, \quad (2.11)$$

so that

$$D^e \approx R^e E^e R^e T \quad \text{and} \quad W^e \approx R^e R^e T. \quad (2.12)$$

We emphasize that, while we make the assumption of small elastic strains, we still allow for elastic deformation to occur and for elastic power to be expended.

2.3. Kinematics of plastic flow

Next, we make the kinematical assumption that plastic flow is incompressible,

$$J^p = 1 \quad \text{and} \quad \text{tr} L^p = \text{tr} D^p = 0, \quad (2.13)$$

so that $J^p = J$. This common assumption (Aranson and Tsimring, 2002; Mohan et al., 2002; Kamrin and Bazant, 2007; Jenkins and Berzi, 2010; Kamrin, 2010; Henann and Kamrin, 2013) is in keeping with our interest in modeling steady granular flows and provides a considerable simplification. We acknowledge that transient plastic volumetric dilatation or compaction can be an important effect in granular deformation (Schoefield and Wroth, 1968; Anand and Gu, 2000) and note that our subsequent theoretical development does not preclude the inclusion of plastic volume changes. This assumption coupled with that of the previous section ($J^e \approx 1$) implies that the volume of material elements in the referential, intermediate, and spatial
configurations are all approximately equal, and henceforth, we will not distinguish between referential, intermediate, and spatial volume densities.

Further, we will focus on an isotropic description and make the standard assumption that plastic flow is irrotational (see the discussion in Gurtin and Anand (2005a)) so that

$$ W^p = 0. $$

(2.14)


Next, let

$$ \dot{\gamma}^p = \sqrt{2|D^p|} \geq 0 $$

(2.15)

denote the equivalent shear plastic strain rate. Then, whenever $\dot{\gamma}^p \neq 0$,

$$ N^p = \sqrt{2 D^p / \dot{\gamma}^p}, \quad \text{with} \quad \text{tr} N^p = 0, $$

(2.16)

defines the plastic flow direction, and we may write the plastic stretching as

$$ D^p = \frac{1}{\sqrt{2}} \dot{\gamma}^p N^p. $$

(2.17)

Finally, using (2.2), (2.3), (2.10), (2.14), and (2.17), we write for future use

$$ \text{grad} v = L^p + \frac{1}{\sqrt{2}} \dot{\gamma}^p R^p N^p R^p \cdot $$

(2.18)

2.4. Granular fluidity

We introduce a strictly non-negative state parameter, $g \geq 0$, with units of (1/time), which we refer to as the granular fluidity (Kamrin and Koval, 2012; Henann and Kamrin, 2013). The granular fluidity is a continuum-level quantity representing the susceptibility of a granular element to flow. More precisely, based on the microscopic derivation of Bocquet et al. (2009), the granular fluidity may be thought of as the rate of plastic events. In their analysis, plastic events occur over a characteristic rearrangement time-scale, $t_r$. Accordingly, we introduce a dimensionless granular fluidity $g = t_c g$ and proceed to develop a theory that depends upon this field, described spatially as $g(x, t)$. Multiple interpretations of the field $g$ are possible; however, we emphasize that a concrete definition of $g$ is not necessary to the development of a continuum theory. We give an interpretation of the dimensionless fluidity field as related to a flow carrier density – a kinematical variable – in Appendix A.

To simplify notation, we drop the overbar from the dimensionless granular fluidity in the analysis of Sections 2 and 3 and denote its spatial gradient as $g = \text{grad} g$. For future use, we note that the material time derivative of $g$ is given by

$$ g = \text{grad} g - L^\top g. $$

(2.19)

2.5. Internal and external expenditures of power

The application of the principle of virtual power requires an identification of the sources of externally and internally-expended power. Since the Eqs. (1.1)–(1.5) are given in the deformed space, we define power expenditures spatially as well. Consider an arbitrary part $P_i$ of the deformed body $B_i$ with $n$ the outward unit normal on the boundary $\partial P_i$ of $P_i$. We take the external power expended on $P_i$ to be due to

(a) a surface traction $t(n)$, measured per unit area in the deformed body, that expends power conjugate to the velocity $v$ on the boundary $\partial P_i$, (b) a body force $b$, measured per unit volume – presumed to account for inertia, granted the underlying frame is inertial, so that $b = b_0 - \rho v$, where $b_0$ is the non-inertial body force and $\rho$ is the mass density of the granular media – that also expends power over $v$ in $P_i$, and (c) a scalar microscopic traction $\zeta(n)$ that expends power conjugate to $g$ on the boundary $\partial P_i$, so that the external power is given by

$$ W_{\text{ext}}(P_i) = \int_{\partial P_i} t(n) \cdot v \, da + \int_{P_i} b \cdot v \, dv + \int_{\partial P_i} \zeta(n) g \, da. $$

(2.20)

Next, we consider the manner in which internal power is expended. This requires identifying the rate-like fields over which power is expended and introducing appropriate power-conjugate stress measures. We assume the following internal power expenditures:
(a) an elastic stress $\mathbf{T}^e$ expending power conjugate to the elastic velocity gradient $\mathbf{L}^e$,
(b) a non-negative, scalar plastic stress $\pi$ expending power conjugate to the equivalent shear plastic strain rate $\dot{\gamma}^p$,
(c) a scalar microforce $m$ expending power conjugate to $\dot{\mathbf{g}}$, and
(d) a vector microforce $\zeta$ expending power conjugate to $\mathbf{g}$.

so that the internal power is given by

$$\mathcal{W}_{\text{int}}(P_t) = \int_{P_t} (\mathbf{T}^e : \mathbf{L}^e + \pi \dot{\gamma}^p + m \dot{\mathbf{g}} \cdot \dot{\mathbf{g}}) dv. \quad (2.21)$$

We invoke the physical requirement that the internal power $\mathcal{W}_{\text{int}}(P_t)$ be invariant under a change in frame. A detailed discussion regarding changes in frame is given in Appendix B. An important consequence of this requirement is that the tensorial quantity $(\mathbf{T}^e - \mathbf{g} \otimes \zeta)$ is symmetric (see (B.11)),

$$(\mathbf{T}^e - \mathbf{g} \otimes \zeta) = (\mathbf{T}^e - \mathbf{g} \otimes \zeta)^T. \quad (2.22)$$

2.6. Principle of virtual power

Following the work of Gurtin (Gurtin, 1996, 2002; Gurtin and Anand, 2005a,b,c), we derive the force balances of our theory using the principle of virtual power. We assume that, at some fixed time, the kinematic fields $- \mathbf{F}, \mathbf{R}, \mathbf{F}^p, \mathbf{N}, \mathbf{g}$, and $\hat{\mathbf{g}}$ are known, and consider the rate-like fields $\mathbf{v}, \mathbf{L}^e, \dot{\gamma}^p, \hat{\mathbf{g}}$, and $\mathbf{g}$ as virtual velocities. We denote the virtual fields by $\hat{\mathbf{v}}, \hat{\mathbf{L}}^e, \dot{\gamma}^p, \hat{\mathbf{g}}$, and $\hat{\mathbf{g}}$ to distinguish them from the real rate-like fields and require that they be consistent with the constraints (2.18) and (2.19):

$$\begin{align*}
\{ & \text{grad} \hat{\mathbf{v}} = \mathbf{L}^e + \frac{1}{\sqrt{2}} \dot{\gamma}^p \mathbf{R}^p \mathbf{N}^p \mathbf{R}^p, \\
& \hat{\mathbf{g}} = \text{grad} \hat{\mathbf{g}} - (\text{grad} \hat{\mathbf{v}})^T \mathbf{g} \}.
\end{align*} \quad (2.23)$$

We denote a generalized virtual velocity as a list

$$\mathcal{V} = (\hat{\mathbf{v}}, \hat{\mathbf{L}}^e, \dot{\gamma}^p, \hat{\mathbf{g}}, \mathbf{g}). \quad (2.24)$$

consistent with (2.23), and write $\mathcal{W}_{\text{ext}}(P_t, \mathcal{V})$ and $\mathcal{W}_{\text{int}}(P_t, \mathcal{V})$ for the external and internal expenditures of power, respectively, when the real rate-like fields are replaced by their virtual counterparts. We then postulate a principle of virtual power requiring that, given any generalized virtual velocity $\mathcal{V}$ and any part $P_t$, the corresponding internal and external virtual powers are balanced:

$$\mathcal{W}_{\text{ext}}(P_t, \mathcal{V}) = \mathcal{W}_{\text{int}}(P_t, \mathcal{V}). \quad (2.25)$$

2.7. Force balances

In applying the virtual balance (2.25), we may choose any $\mathcal{V}$ consistent with the constraints (2.23). The list $\mathcal{V}$ contains five virtual velocities subject to two constraints, so that three virtual velocities may be prescribed independently, implying that there are three independent force balances. To deduce each force balance, we will (i) take two virtual velocities to be zero, (ii) apply the constraints, and (iii) derive the force balance by localizing the corresponding reduced power balance.

2.7.1. Macroforce balance

We first consider a generalized virtual velocity, in which $\dot{\gamma}^p = 0$, $\hat{\mathbf{g}} = 0$, and the virtual field $\hat{\mathbf{v}}$ is chosen arbitrarily. The constraints (2.23) then dictate that $\mathbf{L}^e = \text{grad} \hat{\mathbf{v}}$ and $\hat{\mathbf{g}} = -(\text{grad} \hat{\mathbf{v}})^T \mathbf{g}$. The power balance (2.25) then takes the form

$$\int_{P_t} \mathbf{t}(n) \cdot \hat{\mathbf{v}} \, da + \int_{P_t} \mathbf{b} \cdot \hat{\mathbf{v}} \, dv = \int_{P_t} (\mathbf{T}^e - \mathbf{g} \otimes \zeta) : \text{grad} \hat{\mathbf{v}} \, dv. \quad (2.26)$$

Applying the divergence theorem to (2.26) yields

$$\int_{P_t} [\mathbf{t}(n) - (\mathbf{T}^e - \mathbf{g} \otimes \zeta)n] \cdot \hat{\mathbf{v}} \, da + \int_{P_t} [\text{div}(\mathbf{T}^e - \mathbf{g} \otimes \zeta) + \mathbf{b}] \cdot \hat{\mathbf{v}} \, dv = 0.$$

Since this relation must hold for all $P_t$ and all $\mathbf{v}$, standard variational arguments yield the traction condition

$$\mathbf{t}(n) = (\mathbf{T}^e - \mathbf{g} \otimes \zeta)n,$$ \quad (2.27)

and the local macroforce balance

$$\text{div}(\mathbf{T}^e - \mathbf{g} \otimes \zeta) + \mathbf{b} = 0,$$ \quad (2.28)

respectively. Thus, the stress
\[ T = (T^e - \mathbf{g} \otimes \zeta) \]  
(2.29)
represents the classical, symmetric Cauchy stress, and (2.28) and (2.22) represent the local macroscopic force and moment balances in the spatial configuration.

### 2.7.2. First microforce balance

Next, consider a generalized virtual velocity with \( \mathbf{v} = 0 \) and \( \mathbf{g} = 0 \), and choose the virtual field \( \tilde{\mathbf{g}}^p \) arbitrarily, so that by (2.23),

\[ \mathbf{L}^* = -\frac{1}{\sqrt{2}} \tilde{\mathbf{g}}^p \mathbf{R}^p \mathbf{R}^{*T} \quad \text{and} \quad \mathbf{g} = 0. \]

Thus,

\[ \mathbf{T}^* : \mathbf{L}^* = -\left( \frac{1}{\sqrt{2}} \left( \mathbf{R}^{*T} \mathbf{T}^e \mathbf{R}^e \right) : \mathbf{N}^p \right) \tilde{\mathbf{g}}^p, \]

which motivates the definition of a Mandel stress

\[ \mathbf{M}^e = \mathbf{R}^{*T} \mathbf{T}^e \mathbf{R}^e. \]

Because we have chosen \( \mathbf{v} = 0 \), the external power vanishes, so that the power balance (2.25) implies

\[ W_{\text{int}}(P_t, V) = \int_{\Omega} \left( \pi - \frac{1}{\sqrt{2}} \mathbf{M}_0^e : \mathbf{N}^p \right) \tilde{\mathbf{g}}^p \, d\mathbf{v} = 0, \]

where we have noted the deviatoric nature of \( \mathbf{N}^p \). Since this must be satisfied for all \( P_t \) and all \( \tilde{\mathbf{g}}^p \), the standard argument yields the first microforce balance

\[ \frac{1}{\sqrt{2}} \mathbf{M}_0^e : \mathbf{N}^p = \pi. \]

### 2.7.3. Second microforce balance

Finally, consider a generalized virtual velocity with \( \mathbf{v} = 0 \) and \( \tilde{\mathbf{g}}^p = 0 \), so that by (2.23), \( \mathbf{L}^e = 0 \) and \( \mathbf{g} = \text{grad} \tilde{\mathbf{g}} \), and choose \( \tilde{\mathbf{g}} \) arbitrarily. Then, (2.25) takes the form

\[ \int_{\partial \Omega} \zeta(\mathbf{n}) \tilde{\mathbf{g}} \, d\mathbf{a} = \int_{\Omega} \left( \mathbf{m} \tilde{\mathbf{g}} + \zeta \cdot \text{grad} \tilde{\mathbf{g}} \right) \, d\mathbf{v}. \]

Using the divergence theorem, (2.34) yields

\[ \int_{\partial \Omega} \left( \zeta(\mathbf{n}) - \zeta \cdot \mathbf{n} \right) \tilde{\mathbf{g}} \, d\mathbf{a} + \int_{\Omega} (\text{div} \zeta - m) \tilde{\mathbf{g}} \, d\mathbf{v} = 0, \]

and the standard argument yields the microscopic traction condition

\[ \zeta(\mathbf{n}) = \zeta \cdot \mathbf{n} \]

and the second microforce balance

\[ \text{div} \zeta - m = 0. \]

### 2.8. Free-energy imbalance

Granular materials are athermal, and as such, we consider a purely mechanical theory in which energy does not conduct as heat. In such a theory, the second law of thermodynamics requires that the time rate of change of the free energy in any part \( P_t \) be less than or equal to the power expended on \( P_t \), leading to the inequality,

\[ \int_{P_t} \psi \, d\mathbf{v} \leq W_{\text{ext}}(P_t) = W_{\text{int}}(P_t), \]

with \( \psi \) the Helmholtz free energy, measured per unit volume. Since we have assumed negligible change in volume between the referential and spatial configurations, and since \( P_t \) deforms with the body,

\[ \int_{\Omega} \psi \, d\mathbf{v} = \int_{\Omega} \psi \, d\mathbf{v}. \]

Thus, since \( P_t \) is arbitrary, we may use (2.21) to localize (2.38), resulting in the local free-energy imbalance
\[ \dot{\psi} - T^e : L^e - \pi^e \rho - \sigma \dot{g} - \zeta : \dot{g} \leq 0. \]  
(2.39)

Using (2.11) and (2.31), we rewrite the elastic power as
\[ T^e : L^e = T^e : (R^e E^e R^e) + T^e : (R^e R^e) = (R^e T^e R^e) : \dot{E}^e + T^e : (R^e R^e) = \dot{M}^e : \dot{E}^e + T^e : (R^e R^e). \]  
(2.40)

Use of (2.40) in (2.39) allows us to express the free-energy imbalance as
\[ \dot{\psi} - M^e : \dot{E}^e - T^e : (R^e R^e) - \pi^e \rho - \sigma \dot{g} - \zeta : \dot{g} \leq 0. \]  
(2.41)

2.9. Free energy imbalance based on the Gibbs free energy

We will find it convenient to develop our theory using the stress as the independent variable. Accordingly, we introduce the Gibbs free energy \( \phi \), defined through the Legendre transform
\[ \phi = \psi - M^e : \dot{E}^e. \]  
(2.42)

Differentiating (2.42),
\[ \dot{\phi} = \dot{\psi} - M^e : \dot{E}^e - \dot{E}^e : \dot{M}^e, \]  
(2.43)

we may write the free energy imbalance (2.41) as
\[ \dot{\phi} + \dot{E}^e : \dot{M}^e - T^e : (R^e R^e) - \pi^e \rho - \sigma \dot{g} - \zeta : \dot{g} \leq 0. \]  
(2.44)

3. Constitutive theory

3.1. Energetic constitutive equations

Guided by the free energy imbalance (2.44), we consider constitutive equations for the Gibbs free energy \( \phi \), the elastic strain \( E^e \), and the microstress \( \zeta \) of the form:
\[ \phi = \phi(M^e, g, g) \]
\[ E^e = E^e(M^e, g, g) \]
\[ \zeta = \zeta(M^e, g, g). \]  
(3.1)

Then, from (3.1),
\[ \dot{\phi} = \frac{\partial \phi}{\partial M^e} : \dot{M}^e + \sigma_{en} \dot{g} + \frac{\partial \phi}{\partial g} : \dot{g}. \]  
(3.2)

where
\[ \sigma_{en} = \frac{\partial \phi}{\partial g}, \]  
(3.3)

which we refer to as the energetic microstress. Satisfaction of the free energy imbalance (2.44) requires that
\[ \left( \frac{\partial \phi}{\partial M^e} + \dot{E}^e \right) : \dot{M}^e - T^e : (R^e R^e) - \dot{\pi}^e \rho - \sigma_{dis} \dot{g} + \left( \frac{\partial \phi}{\partial g} - \zeta \right) : \dot{g} \leq 0, \]  
(3.4)

where we have defined a dissipative microstress \( \sigma_{dis} \) via the relation
\[ \sigma_{dis} = \sigma - \sigma_{en}. \]  
(3.5)

The inequality (3.4) is required to hold for all values of \( M^e \) and \( g \). Therefore, we are led to the thermodynamic restriction that the free energy \( \phi \) determines the strain \( E^e \) and the microstress \( \zeta \) through the “state relations”
\[ \dot{E}^e = -\frac{\partial \phi}{\partial M^e} \quad \text{and} \quad \dot{\zeta} = \frac{\partial \phi}{\partial g} \]  
(3.6)

3.1.1. Consequences of material frame indifference on the free energy

The fields \( \phi, M^e, \) and \( g \) are invariant under a change in frame (see Appendix B), while \( g \) transforms as \( Qg \) (see (B.3)). Hence the constitutive equation for \( \phi \) must satisfy
\[ \dot{\phi}(M^e, g, g) = \dot{\phi}(M^e, g, Qg). \]  
(3.7)
for all proper, orthogonal rotations $Q$. The constitutive equation for $\phi$ may then be represented as

$$\phi = \phi(M^e, g, |g|),$$

and therefore, by (3.6)2, the vectors $\zeta$ and $g$ are collinear. This observation, along with (2.22), implies that $T^e$ is symmetric and that

$$T^e : (R^eR^e) = 0.$$  

(3.9)

3.1.2. Isotropic free energy

As a consequence of isotropy, the free energy function $\dot{\phi}(M^e, g, |g|)$ is an isotropic function of its arguments. Hence, the free energy function has the representation

$$\dot{\phi}(M^e, g, |g|) = \tilde{\phi}(I_{M^e}, g, |g|),$$

(3.10)

with $I_{M^e}$ a list of invariants of $M^e$. A list of independent invariants are

$$I_{M^e} = \{\bar{p}, \bar{\tau}, \det M^e\},$$

where $\bar{p}$ is the mean normal pressure and $\bar{\tau}$ is the equivalent shear stress, given by

$$\bar{p} = -\frac{1}{3} \text{tr}M^e \quad \text{and} \quad \bar{\tau} = \frac{1}{\sqrt{2}} |M^e|,$$

(3.11)

respectively. Here, we neglect dependence of the free energy function on the determinant of $M^e$, and take the free energy to be given by

$$\phi = \tilde{\phi}(\bar{p}, \bar{\tau}, g, |g|).$$

(3.12)

3.1.3. Separability assumption for the free energy

We consider a separable free energy of the form

$$\phi(\bar{p}, \bar{\tau}, g, |g|) = \phi^{(\text{e})}(\bar{p}, \bar{\tau}) + \phi^{(\text{g})}(\bar{p}, \bar{\tau}, g) + \phi^{(\text{s})}(|g|).$$

(3.13)

where $\phi^{(\text{e})}$ is the contribution to the change in free energy due to elastic deformation, $\phi^{(\text{g})}$ is a coarse-grain Ginzburg–Landau-type free energy, and $\phi^{(\text{s})}$ is a nonlocal contribution to the free energy, so that by (3.3) and (3.6), we have

$$F^e = -\frac{\partial \phi^{(\text{e})}(\bar{p}, \bar{\tau})}{\partial M^e} - \frac{\partial \phi^{(\text{g})}(\bar{p}, \bar{\tau}, g)}{\partial M^e}, \quad \sigma_{\text{en}} = \frac{\partial \phi^{(\text{g})}(\bar{p}, \bar{\tau}, g)}{\partial g}, \quad \text{and} \quad \zeta = \frac{\partial \phi^{(\text{s})}(|g|)}{\partial g}.$$  

(3.14)

3.2. Dissipative constitutive equations

Use of (3.6) and (3.9) in (3.4) leaves us with the reduced dissipation inequality

$$D = \pi \ddot{\gamma}^p + \sigma_{\text{dis}} \dot{g} \geq 0,$$  

(3.15)

where $D$ is the dissipation rate per unit volume. As a simple constitutive equation for the scalar plastic stress $\pi$ and dissipative microstress $\sigma_{\text{dis}}$, we assume that

$$\pi = \pi(\bar{p}, g, \dot{\gamma}^p), \quad \sigma_{\text{dis}} = \sigma_{\text{dis}}(\bar{p}, \bar{\tau}, g, \dot{g}).$$  

(3.16)

We assume that each of the two terms in (3.15) individually satisfy

$$\pi \ddot{\gamma}^p \geq 0, \quad \sigma_{\text{dis}} \dot{g} \geq 0.$$  

(3.17)

(3.18)

We further assume that plastic flow in the material is strongly dissipative in the sense that

$$\pi \ddot{\gamma}^p > 0 \quad \text{for} \quad \dot{\gamma}^p > 0.$$  

(3.19)

Thus, the constitutive equation $\pi$ must obey

$$\pi(\bar{p}, g, \dot{\gamma}^p) > 0 \quad \text{whenever} \quad \dot{\gamma}^p > 0.$$  

(3.20)

Further, the most general form of $\sigma_{\text{dis}}$ consistent with (3.18) is

$$\sigma_{\text{dis}}(\bar{p}, \bar{\tau}, g, \dot{g}) = \beta(\bar{p}, \bar{\tau}, g, \dot{g}) \dot{g}.$$  

(3.21)
where
\[
\hat{\beta}(p, \tau, g, \tilde{g}) \geq 0
\]  
(3.22)
is a non-negative constitutive modulus with units of viscosity. We assume henceforth that the functions \(\hat{\tau}\) and \(\hat{\beta}\) satisfy (3.20) and (3.22), respectively, so that the dissipation inequality (3.15) is not violated.

3.2.1. Flow rule for isotropic materials
We adopt the classical codirectionality hypothesis, relating the direction of plastic flow \(N^p\) to the Mandel stress deviator \(M_e^0\) by
\[
N^p = \frac{M_e^0}{|M_e^0|}.
\]  
(3.23)
We make this choice on pragmatic grounds, as it provides a reasonable description of experimental data in a straightforward manner (Jop et al., 2006; Rycroft et al., 2009; Kamrin, 2010; Henann and Kamrin, 2013). One may adopt a different specification of the direction of plastic flow, such as a double-shearing framework (Spencer, 1964; Mehrabadi and Cowin, 1978; Anand and Gu, 2000), without substantially affecting the framework derived here.

The first microforce balance (2.33) and the constitutive relation (3.16), along with (3.11) and (3.23), then give the scalar plastic flow rule
\[
\hat{\beta} = \hat{\beta}(p, g, \tilde{g}).
\]  
(3.24)
Further, the plastic stretching \(D^p\) in (2.17) may be written as
\[
D^p = \frac{\hat{\beta}}{2\tau} \frac{M_e^0}{|M_e^0|} \hat{\beta} \geq 0,
\]  
(3.25)
with \(\hat{\beta}\), whenever it is greater than zero, being determined by solving the scalar flow rule (3.24).

3.3. Partial differential equation for the granular fluidity
Upon using the constitutive Eqs. (3.14) and (3.21) in the second microforce balance (2.37), we obtain the following differential relation
\[
\hat{\beta} \frac{\partial g}{\partial t} = \text{div} \left( \frac{\partial \phi^{(B)}}{\partial g} - \frac{\partial \phi^{(S)}}{\partial g} \right),
\]  
(3.26)
which will serve as a Ginzburg–Landau-like partial differential equation for the granular fluidity state variable \(g\).

4. Specialization of the constitutive theory
We now specialize the general constitutive theory described in the preceding section to granular materials. Specifically, our interest is in developing a specialized theory appropriate for describing dense granular flows at steady-state. Based on physical grounds for granular materials, we enforce the standard requirement that the pressure \(p\) be positive, and hence that the domains of the constitutive functions be restricted to \(p > 0\).

To this point, the theory has been developed using the dimensionless granular fluidity \(\tilde{g} = t_g g\) with \(t_g\) the (constant) rearrangement time-scale and the overbar dropped for notational economy. For clarity in specializing the theory, for the remainder of the paper, we absorb the constant time-scale \(t_g\) back into the fluidity, returning \(g\) to its commonly-treated dimensional form with units of \((1/\text{time})\).

4.1. Energetic constitutive equations
We first address the three free energy functions \(\phi^{(e)}, \phi^{(R)}, \text{ and } \phi^{(S)}:\n
1. The mean-field elastic response of a granular medium actually has little to no bearing on velocities fields or forces arising in steady granular flows.\(^4\) Therefore, we leave the exact form of the elastic free energy unspecified, i.e. \(\phi^{(e)} = \phi^{(e)}(\tilde{p}, \tilde{\tau})\). Options include a quadratic form of the free energy with constant bulk and shear moduli or a more complex form in which the bulk and shear moduli depend on pressure as in Jiang–Liu granular elasticity (Jiang and Liu, 2003).
2. We consider a granular material made up of grains characterized by a mean diameter \(d\) and made of a material with density \(\rho_s\). Consistent with Bagnold scaling (Bagnold, 1954), we introduce the stress ratio,
\[
\mu = \frac{\tilde{\tau}}{\tilde{p}}.
\]  
(4.1)

\(^4\) The mean-field elastic response refers to the averaged macro-scale response and is distinct from the micro-scale elastic interactions between grains.
and take the coarse-grain free energy to be

\[ \phi^{(c)} = C \rho_s d^2 \left[ \frac{1}{2} (\mu_s - \mu) g^2 + \frac{1}{3} b \sqrt{\frac{\rho_s d^2}{p}} \kappa g^2 \right], \tag{4.2} \]

where \( \mu_s > 0 \) is a dimensionless critical (static yield) value of the stress ratio, \( b > 0 \) is a dimensionless material parameter, and \( C > 0 \) is a dimensionless prefactor. The character of \( \phi^{(c)} \) is different depending on the sign of \( (\mu_s - \mu) \), see Fig. 1. When \( \mu < \mu_s \), a single stable equilibrium point (local minimum) exists at \( g = 0 \). For the case of \( \mu > \mu_s \), the equilibrium point \( g = 0 \) becomes unstable (local maximum) and a new stable equilibrium point appears for \( g > 0 \). In this way, the free energy \( \phi^{(c)} \) describes the phase transition between solid-like, non-flowing states, \( g = 0 \), and fluidized states, \( g > 0 \).

3. Finally, the nonlocal free energy is

\[ \phi^{(nl)} = \frac{1}{2} CA^2 \rho_s d^4 |g|^2, \tag{4.3} \]

where \( A > 0 \) is another dimensionless material parameter.

Therefore the total Gibbs free energy is given by

\[ \phi = \phi^{(c)} + C \rho_s d^2 \left[ \frac{1}{2} (\mu_s - \mu) g^2 + \frac{1}{3} b \sqrt{\frac{\rho_s d^2}{p}} \kappa g^2 \right] + \frac{1}{2} CA^2 \rho_s d^4 |g|^2. \tag{4.4} \]

Next, we make the assumption that the elastic free energy is dominant. That is to say, we assume that the magnitude of the free energy due to elastic deformation, \(|\phi^{(e)}|\), is much greater than the other contributions, \(|\phi^{(nl)}|\) and \(|\phi^{(c)}|\).

- Equivalently, we assert that the parameter \( C \) is a small number, and in subsequent steps, we truncate at leading order where appropriate.

Under this assumption, straightforward calculations involving (3.14), (4.2), and (4.3) yield the energetic constitutive equations

\[
\begin{align*}
E^e &= \frac{\partial \phi^{(e)}}{\partial \mathbf{M}^e}, \\
\sigma_{en} &= \frac{1}{t_e} \frac{\partial \phi^{(e)}}{\partial g} = C \rho_s d^2 \left[ (\mu_s - \mu) g + b \sqrt{\frac{\rho_s d^2}{p}} \kappa g^2 \right], \\
\zeta &= \frac{1}{t_e} \frac{\partial \phi^{(e)}}{\partial g} = CA^2 \rho_s d^4. \tag{4.5}
\end{align*}
\]

where the relaxation time \( t_e \) is now explicitly included since we use the dimensional form of \( g \). In (4.5)_1, we have truncated the \((- \partial \phi^{(e)}/\partial \mathbf{M}^e)\) term in (3.14)_1 due to the smallness of \( C \). Our scaling assumption for the free energy also allows the Cauchy stress (2.29) to be truncated as

\[ \mathbf{T} = \mathbf{R}^e \mathbf{M}^e \mathbf{R}^{eT}, \tag{4.6} \]

where we have made use of (2.31), and \( \mathbf{M}^e \) is given implicitly through (4.5)_1. The validity of truncating at leading order hinges on our ability to show that the terms multiplied by \( C \) are bounded, which we address in Appendix C.

Fig. 1. Schematic of the coarse-grain Ginzburg–Landau free energy \( \phi^{(c)} \) as a function of the granular fluidity \( g \) for stress ratios both below and above the critical stress ratio \( \mu_s \).
4.2. Dissipative constitutive equations

For the scalar flow rule (3.24), we take the simple relation
\[ \hat{\gamma} = \frac{\dot{\gamma}}{g} \]  
(4.7)

Here, we have adhered to the notion that the stress-scale in a granular material is set by the pressure. This choice of constitutive equation makes concrete the role of the granular fluidity \( g \). By virtue of (4.7), \( g \) acts as an inverse-viscosity-like quantity relating the stress ratio \( \mu \) and the equivalent shear plastic strain rate \( \dot{\gamma} \). That is to say, by (4.7), we have that \( \mu = \frac{\dot{\gamma}}{g} \). Further, since \( g \geq 0 \) and \( \dot{\gamma} > 0 \), the requirement (3.20) is satisfied. From (3.25), the plastic stretching \( \mathbf{D}^p \) is given by
\[ \mathbf{D}^p = \frac{1}{2} \left( \frac{g}{\dot{\gamma}} \right) \mathbf{M}^p_0 \quad \text{so that} \quad \mathbf{M}^p_0 = 2 \left( \frac{p}{g} \right) \mathbf{D}^p. \]  
(4.8)

Finally, the constitutive equation \( \dot{\gamma} \) allows for the modeling of transient effects in the evolution of \( g \). In the interest of simplicity, we take
\[ \dot{\gamma} = \frac{C \rho_s d^2}{t_r^2} - t_0, \]  
(4.9)

where \( t_0 > 0 \) is a constant time-scale associated with the dynamics of \( g \), distinct from the rearrangement time-scale \( t_r \). Here, we have assumed that \( \dot{\gamma} \), like \( \phi^{(r)} \) and \( \phi^{(b)} \), is of order \( C \). We then have
\[ \tau_{\text{dis}} = \frac{C \rho_s d^2}{t_r} t_0 g, \]  
(4.10)

and hence
\[ \tau = \tau_{\text{en}} + \tau_{\text{dis}} = \frac{C \rho_s d^2}{t_r} \left[ (\mu_s - \mu) g + b \sqrt{\frac{\rho_s d^2}{p}} \mu g^2 \right] + \frac{C \rho_s d^2}{t_r} t_0 g. \]  
(4.11)

4.3. Partial differential equation for the granular fluidity

Since \( \phi^{(r)} \) does not appear in (3.26), the contributions due to \( \phi^{(b)} \) and \( \phi^{(b)} \) are of leading order and cannot be neglected in the context of this equation. Upon use of (4.2), (4.3), and (4.9) in (3.26), we obtain the following differential relation
\[ \frac{C \rho_s d^2}{t_r} t_0 g = \text{div} \left( \frac{CA^2 \rho_s d^2}{t_r} \mathbf{g} \right) - \frac{C \rho_s d^2}{t_r} \left[ (\mu_s - \mu) g + b \sqrt{\frac{\rho_s d^2}{p}} \mu g^2 \right]. \]  
(4.12)

Denoting the Laplacian operator as \( \nabla^2 \) = div(grad(\cdot)), allows (4.12) to be written as
\[ t_0 g = A^2 d^2 \nabla^2 g - (\mu_s - \mu) g - b \sqrt{\frac{\rho_s d^2}{p}} \mu g^2. \]  
(4.13)

4.3.1. A steady form of the partial differential equation for the granular fluidity

We turn to deriving a steady-state form of (4.13) using a sequence of assumptions common to Ginzburg–Landau analysis. The reason this task is more complex than merely setting \( g = 0 \) in (4.13) is because the resulting equation may permit multiple solutions (see Fig. 1), whereas we are only interested in the stable solution, corresponding to the solution of (4.13) in the long-time limit.

In the absence of spatial gradients in \( g \), the solution of (4.13) is denoted as the local solution \( g_{\text{loc}} \). It satisfies
\[ t_0 g_{\text{loc}} = -(\mu_s - \mu) g_{\text{loc}} - b \sqrt{\frac{\rho_s d^2}{p}} \mu g_{\text{loc}}^2. \]  
(4.14)

As discussed in Section 4.1, when \( \mu < \mu_s \), the only solution is \( g_{\text{loc}} = 0 \) (given the positivity of \( b \) and \( t_0 \)), and it is stable. However, when \( \mu > \mu_s \), two physical solutions exist (with non-negative fluidity). While the zero solution is still one of them, it is an unstable fixed point (see Fig. 1). The system instead approaches the stable solution, \( g_{\text{loc}} = \sqrt{\frac{b}{\rho_s d^2} (\mu - \mu_s)} / (b \mu) \). Putting together the two stable solutions, we arrive at a general form for stable, steady-state fluidity in the absence of fluidity gradients:
\[ g_{\text{loc}} = H(\mu - \mu_s) \sqrt{\frac{b}{\rho_s d^2} \frac{\mu - \mu_s}{b \mu}}, \]  
(4.15)
where $H$ is the Heaviside function. We note that this relation yields the inertial scaling behavior in the homogeneous flow case (1.4), i.e.

$$
\mathcal{J}_{\text{loc}} = g_{\text{loc}} \mu = H(\mu - \mu_s) \sqrt{\frac{p}{\rho} \frac{\mu - \mu_s}{b}}.
$$

(4.16)

Next, we study steady-state settings ($g = 0$), allowing for spatial gradients but limiting our analysis to small deviations of $g$ from $g_{\text{loc}}$. Accordingly, we let $g = g_{\text{loc}} + \delta$ for some small function $\delta$ and examine the consequences of this condition on the PDE (4.13). Solutions under this assumption, hence, approximate to stable solutions of (4.13). The analysis carries forward in two cases.

**Case 1:** $\mu < \mu_s$. Here, $g_{\text{loc}} = 0$ so $g = \delta$. This leaves us with

$$
A^2 d^2 \nabla^2 \delta = (\mu_s - \mu) \delta + b \sqrt{\frac{\rho d^2}{p} \mu \delta^2}.
$$

(4.17)

By order-of-magnitude arguments, we can neglect the $\delta^2$ term leaving us with

$$
A^2 d^2 \nabla^2 g = (\mu_s - \mu) g.
$$

(4.18)

**Case 2:** $\mu > \mu_s$. In this case, $g_{\text{loc}}$ is non-zero. Utilizing a Taylor approximation, we may write

$$
g^2 = (g_{\text{loc}} + \delta)^2 \approx g_{\text{loc}}^2 + 2g_{\text{loc}} \delta = g_{\text{loc}}^2 + 2g_{\text{loc}}(g - g_{\text{loc}}) = -g_{\text{loc}}^2 + 2g_{\text{loc}} g.
$$

(4.19)

which upon substituting into (4.13) and using the local relation (4.15), gives us

$$
A^2 d^2 \nabla^2 g = (\mu_s - \mu) g + b \sqrt{\frac{\rho d^2}{p} \mu (-g_{\text{loc}}^2 + 2g_{\text{loc}} g)}.
$$

(4.20)

$$
= (\mu_s - \mu) g + (\mu_s - \mu)(g_{\text{loc}} - 2g) = (\mu_s - \mu)(g - g_{\text{loc}}).
$$

Putting together (4.18) and (4.20), we have

$$
A^2 d^2 \nabla^2 g = |\mu - \mu_s| (g - g_{\text{loc}}).
$$

(4.21)

Finally, we introduce length-dimensioned quantity, which we call the cooperativity length,

$$
\xi = \frac{Ad}{\sqrt{|\mu - \mu_s|}},
$$

(4.22)

so that (4.21) may be written in compact form as

$$
\nabla^2 g = \frac{1}{\xi^2} (g - g_{\text{loc}}),
$$

(4.23)

with

$$
g_{\text{loc}}(p, \mu) = H(\mu - \mu_s) \sqrt{\frac{p}{\rho} \frac{\mu - \mu_s}{b\mu}} \quad \text{and} \quad \xi(\mu) = \frac{Ad}{\sqrt{|\mu - \mu_s|}},
$$

(4.24)

a result identical to (1.5) and (1.4).

4.4. Summary of the specialized theory

Finally, we summarize our specialized and reduced model, which is appropriate for the description of steady, dense granular flows.

**Kinematics:** The specialized theory involves the following kinematical fields: $\mathbf{x} = \chi(X, t)$, motion; $\mathbf{F} = \nabla \chi$; $J = \det \mathbf{F} > 0$, deformation gradient; $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$, multiplicative elastic–plastic decomposition of $\mathbf{F}$; $\mathbf{F}^e = \det \mathbf{F}^p = 1$, (constant-volume) plastic distortion; $\mathbf{F}^e, \mathbf{J}^e = \det \mathbf{F}^e > 0$, elastic distortion; $\mathbf{F}^p = \mathbf{R}^p \mathbf{U}^p$, polar decomposition of $\mathbf{F}^e$; and $\mathbf{E} = (1/2)(\mathbf{F}^e \mathbf{F}^e - 1)$, elastic Green strain. We make the restriction that the elastic strains are small in the sense that $|\mathbf{E}^e| \ll 1$, so that $\mathbf{F}^e \approx \mathbf{R}^e, \mathbf{U}^e \approx 1$, and $\mathbf{F}^p \approx 1$. Finally, we introduce a scalar state parameter $g$, with units of $s^{-1}$, referred to as the granular fluidity.

---

5 One drawback of the approximate system is that the stability of the global $g = 0$ solution, representing a completely static body, is determined only by the local stress field, through $g_{\text{loc}}$. Likewise, the approximation is specialized to compute flowing solutions and may lack precision in predicting the size-dependence of static strength.
Cauchy stress: The Cauchy stress is given by
\[ T = R^T M^* R, \]  
where \( M^* \) is the Mandel stress, given implicitly through
\[ E^* = -\frac{\partial \phi^*(p, T)}{\partial M^*}, \]  
where \( \phi^* \) is the elastic Gibbs energy, and
\[ \bar{T} = \frac{1}{\sqrt{2}} |M_0^*|, \quad \bar{p} = -\frac{1}{3} tr M^*, \quad \text{and} \quad \mu = \frac{\bar{T}}{\bar{p}} \]  
are the equivalent shear stress, mean normal pressure, and stress ratio, respectively.

Flow rule: The evolution of \( F^* \) is given by
\[ \dot{F}^* = D^p F^*, \]  
with \( D^p \) given by
\[ D^p = \frac{1}{2} \left( \frac{g}{\bar{p}} \right) M_0^*. \]  

Governing partial differential equations and boundary conditions: The governing partial differential equations consist of the local force balance (2.28),
\[ \text{div} T + b_0 = \rho \dot{v}, \]  
with \( T \) given by (4.25), \( b_0 \) the non-inertial body force per unit volume, and \( \rho \) the mass density of the granular material; and the partial differential equation for the granular fluidity (4.23),
\[ \nabla^2 g = \frac{1}{\zeta^2} (g - g_{\text{loc}}), \]  
with
\[ g_{\text{loc}}(\bar{p}, \mu) = H(\mu - \mu_s) \sqrt{\frac{\bar{p}}{\rho_s d^2} \frac{\mu - \mu_s}{b \mu}} \quad \text{and} \quad \zeta(\mu) = \frac{Ad}{\sqrt{|\mu - \mu_s|}}, \]  
denoting the local fluidity and cooperativity length, respectively, \( \{\mu_s, b, A\} \) dimensionless material parameters, \( d \) the mean grain diameter, and \( \rho_s \) the grain material density.

To specify boundary conditions, we let \( S_u \) and \( S_t \) denote complementary subsurfaces of the boundary \( \partial B \), of the deformed body \( B \) \( (S_u \cup S_t = \partial B_s, S_u \cap S_t = \emptyset) \) where displacements and surface tractions are prescribed, respectively. With \( u(X, t) = \chi(X, t) - X \) denoting the displacement field, the mechanical boundary conditions on \( \partial B \) are given by
\[ u = \bar{u} \text{ on } S_u \quad \text{and} \quad Tn = \bar{t} \text{ on } S_t, \]  
where \( \bar{u} \) and \( \bar{t} \) are prescribed. Similarly, we introduce another set of complementary subsurfaces \( S_\bar{g} \) and \( S_\zeta \) \( (S_\bar{g} \cup S_\zeta = \partial B, S_\bar{g} \cap S_\zeta = \emptyset) \) on which the granular fluidity and the normal component of its gradient are prescribed, respectively. The fluidity boundary conditions on \( \partial B \) are then given by
\[ g = \bar{g} \text{ on } S_\bar{g} \quad \text{and} \quad \text{grad} \cdot n = \bar{\zeta} \text{ on } S_\zeta, \]  
where \( \bar{g} \) and \( \bar{\zeta} \) are prescribed.\(^6\)

5. Conclusion

In this paper, we have formulated a rigorous continuum framework for the nonlocal granular rheology. Importantly, we have identified (i) the (dimensionless) granular fluidity as an energetic, Ginzberg–Landau-type order parameter and (ii) the differential relation for the fluidity as a microforce balance. Our derivation says that the free-energy gains a small contribution (in comparison to the mean-field elastic strain energy) from this new order parameter and its gradient. However, while the effect on the free energy is small, it has a large effect on the plastic flow behavior. Hence, the derived dynamical differential relation (4.13) that governs the balance of the small microforces conjugate to fluidity, ultimately gives rise to a non-negligible modification in the observed flow. Following simplifications to reduce the microforce balance to a form appropriate for steady flows, the form of the differential relation (4.31) and (4.32) is the form shown to successfully describe experimental data (Henann and Kamrin, 2013).

\(^6\) We have absorbed the factor of \( C_2^2 \rho_s d^4 / t_i \) in (4.5) into \( \bar{\zeta} \).
One benefit of this general, rigorous approach is that it provides insight into what the boundary conditions for \( g \) mean in terms of power expended through the boundaries. Additionally, such an approach serves as a basis for adding layers of complexity. For example, we may allow for unsteady evolution of \( g \) by giving an experimentally-motivated constitutive function for \( \beta \) (3.22); we may adopt more complex flow rules than the classical codirectionality flow rule used here (3.23); or we may generalize to dilatant behavior by relaxing the incompressibility constraint (2.13).

We close by pointing out that for steady flows, the elastic strains are small and unchanging. If we consider an infinitely-stiff, i.e. rigid, elastic response, the elastic strains are exactly zero, and we have that \( \mathbf{D}' = \mathbf{0} \) and \( \mathbf{D} = \mathbf{R}' \mathbf{D}' \mathbf{R}'^T \). In this case, the pressure \( P \) is replaced by a Lagrange multiplier \( P \), which enforces the constant-volume constraint. Then, by (4.29) and (4.25), we then have that

\[
\mathbf{T} = -P \mathbf{1} + 2 \left( \frac{P}{g} \right) \mathbf{D},
\]

which, when used in (4.30), gives precisely the boundary-value problem described in Henann and Kamrin (2013).

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Appendix A. A kinematical description of the granular fluidity

In developing a gradient theory, one typically assumes power to be expended over the rate of change of a kinematical quantity as well as its gradient. For example, in gradient plasticity, the kinematical quantity is some measure of the plastic strain (Gurtin and Anand, 2005b; Gurtin and Anand, 2005c; Lele and Anand, 2009; Anand et al., 2012). In the present work, we develop a theory that depends upon gradients of the granular fluidity \( g \), a state parameter with units of \((1/\text{time})\). However, the kinematical nature of \( g \) is not clear. In this appendix, we give a kinematical interpretation of \( g \), justifying a gradient theory based on this state parameter.

To proceed, we build a simple physical picture for viscoplastic flow in granular materials, which leads to the scalar flow rule (4.7), allowing us to connect the fluidity to a kinematical quantity. Consider a representative volume element (RVE) with volume \( V \) at a spatial point \( \mathbf{x} \) in the deformed body \( B_I \). At time \( t \), the RVE contains \( n > 0 \) potential sites for shear transformation. Each site corresponds to a local cluster of grains with average volume \( \Omega_c > 0 \), and during a shear transformation event, each cluster undergoes an average local plastic shear strain of \( \gamma_T > 0 \). Therefore, we may write a volume-averaged shear strain rate at the continuum-scale as

\[
\dot{\gamma}^p = \frac{\Omega_c n}{V} \times \mathcal{R}(\mu),
\]

where \( \mathcal{R}(\mu) \) is a rate factor, depending on the stress ratio \( \mu \) and representing the net rate of forward shear transformations at each site. We define the potential sites for shear transformation per unit volume as

\[
c_i = \frac{n}{V}.
\]

which we refer to as the flow carrier density. Our intuition tells us that the rate factor increases with \( \mu \) and decreases with the characteristic rearrangement time-scale \( \tau \), and for a simplistic model, we take

\[
\mathcal{R}(\mu) = \frac{\mu}{\tau}.
\]

Then, using (A.1), (A.2), and (A.3), we have

\[
\dot{\gamma}^p = \left( \frac{\gamma_T \Omega_c c_i}{\tau} \right) \mu. \tag{A.4}
\]

Upon comparing with (4.7), we recognize the quantity in parentheses as the granular fluidity. We expect the quantities \( \gamma_T, \Omega_c, \) and \( \tau \), to be essentially constants, so that the fluidity and the flow carrier density are linearly related,

\[
g = \left( \frac{\gamma_T \Omega_c}{\tau} \right) c_i. \tag{A.5}
\]

The flow carrier density is an intensive state parameter, which is kinematical in nature and is an appropriate basis for a gradient theory. In Section 2.5, we assumed that power is expended over the rate of change of the dimensionless fluidity,

\[
\dot{g} = t_c g = (\gamma_T \Omega_c) c_i, \tag{A.6}
\]
and its gradient. The simple model presented in this appendix gives this quantity a kinematical interpretation as the dimensionless flow carrier density \( \gamma_\tau \Omega_c c_f \).

### Appendix B. Changes in frame

The principle of material frame-indifference requires that our theory be invariant under changes in frame, i.e. smooth, time-dependent, rigid transformations of the current configuration, defined as

\[
\chi'(X, t) = Q(t) \{ \chi(X, t) - o \} + y(t),
\]

with \( Q(t) \) a rotation, \( y(t) \) a point at each \( t \), and \( o \) a fixed origin (Gurtin et al., 2010). In this appendix, we discuss the manner in which the quantities appearing in the theory transform under a change in frame. Standard arguments involving the kinematical quantities lead to the following transformation rules

\[
\begin{align*}
F' &= QF, \\
F_e' &= QF_e, \\
R_e' &= QR_e, \\
U_e, C_e, \text{ and } E_e \text{ are invariant,} \\
L_e' &= QL_e Q^T + QQ^T, \\
D_e' &= QD_e Q^T, \\
W_e' &= QW_e Q^T + QQ^T, \\
F_p, L_p, D_p, \text{ and } W_p \text{ are invariant.}
\end{align*}
\]

The granular fluidity \( g \) is a scalar field and hence is invariant under a change in frame. Its spatial gradient transforms as

\[
g' = Qg.
\]

and \( g \) transforms as

\[
g' = \dot{Q}g + Q\dot{g}.
\]

In order to deduce transformation rules for stress-related quantities, we invoke the physical requirement that the internal power \( W_{int}(\mathcal{P}_f) \) be invariant under a change in frame, i.e.

\[
W_{int}'(\mathcal{P}_f) = W_{int}(\mathcal{P}_f),
\]

where \( \mathcal{P}_f' \) and \( W_{int}'(\mathcal{P}_f') \) represent the region and the internal power in the new frame. In the new frame, \( \mathcal{P}_f \) transforms rigidly to a region \( \mathcal{P}_f' \), the stresses \( T' \), \( \pi' \), \( \sigma' \), and \( \zeta' \) transform to \( T'^* \), \( \pi'^* \), \( \sigma'^* \), and \( \zeta'^* \), while \( L' \) transforms according to (B.2). Under a change in frame \( W_{int}(\mathcal{P}_f) \) transforms to

\[
W_{int}'(\mathcal{P}_f) = \int_{\mathcal{P}_f} \left[ T'^*: \left( QL_e Q^T + QQ^T \right) + \pi'^* \dot{\rho} + \sigma' \dot{g} + \zeta'^* \left( Qg + \dot{Q}g \right) \right] d\nu
\]

\[
= \int_{\mathcal{P}_f} \left[ \dot{Q} T'^* Q\right]: L_e + T'^*: \left( QQ^T \right) + \pi'^* \dot{\rho} + \sigma' \dot{g} + \zeta'^* \left( Qg + \dot{Q}g \right) \right] d\nu,
\]

where we have utilized the fact that the part \( \mathcal{P}_f \) transforms rigidly. Then (B.5) implies that

\[
\int_{\mathcal{P}_f} \left[ \dot{Q} T'^* Q\right]: L_e + T'^*: \left( QQ^T \right) + \pi'^* \dot{\rho} + \sigma' \dot{g} + \zeta'^* \left( Qg + \dot{Q}g \right) \right] d\nu = \int_{\mathcal{P}_f} \left[ T^*: L_e + \pi \dot{\rho} + \sigma \dot{g} + \zeta \cdot \dot{g} \right] d\nu,
\]

or equivalently, since the part \( \mathcal{P}_f \) is arbitrary,

\[
\left( Q T^* Q\right): L_e + T^*: \left( QQ^T \right) + \pi \dot{\rho} + \sigma \dot{g} + \zeta \cdot \left( Qg + \dot{Q}g \right) = T^*: L_e + \pi \dot{\rho} + \sigma \dot{g} + \zeta \cdot \dot{g}.
\]

Since the change in frame is also arbitrary, we may choose it such that \( Q \) is an arbitrary time-independent rotation, so that \( Q = 0 \). In this case, we find that

\[
\left( T^* - Q T^* Q\right): L_e + (\pi - \pi^* ) \dot{\rho} + (\sigma - \sigma^* ) \dot{g} + (\zeta - Q \zeta^* ) \cdot \dot{g} = 0.
\]

Since this must hold for all \( L_e \), all \( \dot{\rho} \), all \( \dot{g} \), and all \( \zeta \), we find that the stress \( T^* \) transforms according to

\[
T'^* = Q T^* Q^*,
\]

\( \pi \) and \( \sigma \) are invariant, and \( \zeta \) transforms according to

\[
\zeta^* = Q \zeta.
\]
Next, if we assume that $\mathbf{Q} = \mathbf{I}$ at the time in question, so that $\mathbf{Q}$ is an arbitrary skew tensor, we find from (B.7), using (B.8) and (B.9), that

$$\mathbf{T}^0 : \mathbf{Q} + \zeta : \dot{\mathbf{Q}} = (\mathbf{T}^0 + \zeta \otimes \mathbf{g}) : \dot{\mathbf{Q}} = 0,$$

or that the tensorial quantity $(\mathbf{T}^0 + \zeta \otimes \mathbf{g})$ is symmetric,

$$(\mathbf{T}^0 + \zeta \otimes \mathbf{g}) = (\mathbf{T}^0 + \zeta \otimes \mathbf{g})^T.$$

We note that (B.10) implies that the tensorial quantity $(\mathbf{T}^0 - \mathbf{g} \otimes \zeta)$ is also symmetric,

$$(\mathbf{T}^0 - \mathbf{g} \otimes \zeta) = (\mathbf{T}^0 - \mathbf{g} \otimes \zeta)^T.$$

As a consequence of the macroforce balance (2.28), we find that this quantity plays the role of the classical, symmetric Cauchy stress $\mathbf{T}$ (see (2.29)). Due to the transformation rules (B.8), (B.3), and (B.9), the Cauchy stress is frame-indifferent. Due to the transformation rules (B.2) and (B.8), the Mandel stress (2.31) is invariant under a change in frame. Finally, the scalar free energies $\psi$ and $\phi$ are also frame-invariant.

**Appendix C. Revisiting the scaling assumption for the free energy**

In Section 4.1, we assume that the free energies $\phi^{(e)}$ and $\phi^{(s)}$ are far less in magnitude than $\phi^{(c)}$ by taking the parameter $C$ to be a small number. The caveat is that the terms multiplied by $C$ must remain bounded. In this appendix, we examine this requirement and make concrete what it means for $C$ to be “small.” The full form of the elastic relation (3.14), with $\phi^{(s)}$ given by (4.2), is

$$E^s = -\frac{\partial \phi^{(e)}}{\partial \mathbf{M}^e} + \frac{C \rho_s d^2}{p} \left[ \frac{1}{2} \right] \frac{\left( \mathbf{M}^e_s / 2 \mathbf{1} \right)}{2} \frac{1}{6} b \sqrt{\frac{\rho_s d^2}{p} g^3 \left( \mathbf{M}^e_s / 2 \mathbf{1} \right)} - \frac{1}{6} b \sqrt{\frac{\rho_s d^2}{p} g^3 \left( \mathbf{M}^e_s / 2 \mathbf{1} \right)}.$$

As assumed in Section 4.3.1, small deviations of $g$ from $g_{loc}$ are second-order effects, and a reasonable approximation of the order of the terms multiplied by $C$ may be obtained by taking $g \approx g_{loc}$ in the elastic response (C.1). Again, we proceed by considering two cases.

**Case 1:** $\mu > \mu_s$. Since $g_{loc} = 0$ so that $g \approx 0$, the terms multiplied by $C$ will be approximately zero, and our assumption is justified.

**Case 2:** $\mu < \mu_s$. Upon substituting the local relation (4.15) into (C.1), we obtain

$$E^s \approx -\frac{\partial \phi^{(e)}}{\partial \mathbf{M}^e} + \frac{C \rho_s d^2}{p} \left[ \frac{1}{2} g_{loc} \left( \mathbf{M}^e_s / 2 \mathbf{1} \right) \frac{1}{6} b \sqrt{\frac{\rho_s d^2}{p} g_{loc} \left( \mathbf{M}^e_s / 2 \mathbf{1} \right)} - \frac{1}{6} b \sqrt{\frac{\rho_s d^2}{p} g_{loc} \left( \mathbf{M}^e_s / 2 \mathbf{1} \right)}.$$

Clearly, the terms multiplied by $C$ are bounded for $\mu > \mu_s$. Finally, in order to make precise what it means for $C$ to be small, we take that the prefactor to the $(\mu - \mu_s)^2$ term in (C.2), evaluated at $\mu = \mu_s$, is small, that is

$$C \ll \frac{6 b^2 \mu_s^2}{},$$

which limits the magnitude of the contribution of the terms arising from $\phi^{(s)}$ to higher order.

Similarly, the full form of the Cauchy stress (2.29), with $\zeta$ given by (4.5), is

$$\mathbf{T} = \mathbf{R}^e \mathbf{M}^e \mathbf{R}^e \mathbf{C} \frac{\rho_s d^2}{p} \mathbf{g} \otimes \mathbf{g}.$$

To justify truncation of the Cauchy stress to (4.6), it is necessary to establish the boundedness of $\mathbf{g}$. We note the elliptic nature of (4.31) indicates that $\mathbf{g}$ will be bounded so long as $\mathbf{g}$ is finite at the boundaries. We rule out a boundary condition in which $\mathbf{g}$ is infinite, since by (2.36) and (2.20), it would imply an infinite power expenditure. Achieving a formulaic bounding of the parameter $C$ (analogous to (C.3)) to guarantee a particular quantitative limiting behavior between the two terms in (C.4) would require a more detailed analysis and is reserved for future work.

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