Granular materials display a unique set of flow properties, which differentiate them from conventional solids or fluids. When a granular medium is subjected to boundary-driven deformation, it usually forms shear bands, the sizes of which depend upon both the geometry and the grain size [1–7]. Until recently, conventional wisdom was that outside of these shear bands the nominally quiescent material behaved as a solid. However, recent experiments of van Hecke and co-workers [8] established that this is not the case. When an intruder is placed far away from the shear band—or primary flow zone—it will move when any nonzero force is applied to it. In contrast, when there is no primary flow, a distinct yield condition is observed; i.e., a critical applied intruder force must be exceeded to move it. This vanishing of the yield stress in the presence of far-away primary flow is quite remarkable and motivates the notion of a “secondary rheology” to describe the rheological changes occurring at different locations in a granular medium due to the existence of a primary flow. These observations have been corroborated in subsequent works [9,10], establishing secondary rheology as a key open question in granular physics. For specificity, let us consider the following example from the work of Reddy, Forterre, and Pouliquen [9] which summarizes the essential phenomenology. In their work, a cylindrical rod was placed vertically in an annular Couette cell filled with grains. When the inner wall was fixed, they observed that the force applied to the rod had to exceed a critical value $F_c$ in order to move it through the granular medium. However, when the inner wall was rotated and a primary flow consisting of an inner-wall-located shear band was formed, the yield condition vanished everywhere, enabling the rod to creep for any applied force. Importantly, several experimental observations were made about the creep phenomenology. (i) For sufficiently low inner wall speeds, secondary rheology is a rate-independent process; i.e., the rod creep speed is linearly proportional to the inner wall speed. (ii) The rod creep speed increases exponentially with the force applied to the intruder. (iii) The rod creeps faster when it is placed closer to the inner wall shear band.

As noted by Reddy, Forterre, and Pouliquen [9], these observations provide a nontrivial test for continuum models of granular flow. In fact, local approaches to continuum modeling of granular materials, such as inertial granular rheology [11–13] or soil mechanics [14,15], which relate the stress at a point to the local strain, strain rate, or locally evolved state variables, are not equipped to address secondary rheology phenomenology. Recently, we proposed a nonlocal rheology for dense granular flows, capable of quantitatively modeling primary flow fields [16–18], based on emerging concepts from the emulsions community [19,20]. The model was demonstrated to accurately describe flow fields and shear-band widths in numerous granular flow experiments in several different flow configurations [17]. Even so, it was not clear if the same model could address the distinct issue of secondary rheology, where the goal is to describe the variation of rheological properties induced by primary motion rather than the primary motion itself. In this Letter, we provide a decisive answer to this question, and to the outstanding issue of secondary rheology in general, by showing that such phenomena are all obtainable from the nonlocal model.

We begin by summarizing the nonlocal granular rheology. We denote the symmetric strain-rate tensor as \( \dot{\gamma}_{ij} = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i) \) with \( v_i \) the velocity vector and
between the Cauchy stress and the strain rate is a flow rate variable that relates the load intensity to flow. Mathematically, it functions as a field to the model is a scalar state variable.

In a local description of granular flow, the fluidity is consitutively given as a function of the stress, in a manner consistent with Bagnold scaling. Simple dimensional analysis applied to the case of homogeneous simple shear produces a one-to-one relationship between the shear stress ratio and a dimensionless version of the strain rate, which when adopted leads to the following local description of the fluidity:

\[ g_{loc} = \frac{\gamma_{loc}}{\mu} = H(\mu - \mu_s)\sqrt{P/P_a}\frac{d^2[(\mu - \mu_s)/\mu b]}{\xi(\mu)} \]

where \( \mu_s \) is a static yield value, \( b \) is a dimensionless constant characterizing the rate-dependent response of the granular media, and \( H \) is the Heaviside step function.

While the local relation successfully describes homogeneous simple shear, in inhomogeneous flows, significant deviation from this local description is observed, which motivates a nonlocal differential relation for the granular fluidity. We have proposed the following specific functional form:

\[ \nabla^2 g = (1/\xi^2)(g - g_{loc}). \]

where \( \nabla^2(\cdot) \) denotes the Laplacian operator, \( \xi(\mu) = \sqrt{d/\sqrt{\mu - \mu_s}} \) is the stress-dependent cooperativity length, and \( A \) is a dimensionless material parameter called the inertial number.

Next, we apply our nonlocal model to the problem of secondary rheology, by using finite-element calculations in the commercial package ABAQUS/STANDARD. For computational efficiency, we consider a two-dimensional analogue of the experiments of Reddy, Forthery, and Pouliquen [9], pictured in Fig. 1. The geometry is a planar annular shear cell with rough walls at an inner radius \( R_i \) and outer radius \( R_o \). The inner wall is specified to rotate at a fixed rate \( \Omega \), and the outer wall does not rotate but may move radially so as to impart a confining pressure \( P_a \). A circular intruder with diameter \( D \), which we specify to be rigid and frictionless, is located at a distance \( L \) away from the inner wall. Following Ref. [9], we take \( R_i/d = 60 \), \( R_o/d = 180 \), and \( D/d = 2 \), throughout, and consider different values of \( L/d \). The value of outer radius is sufficiently large so as not to affect the calculation results, consistent with experiments. Finally, in our calculations, either the speed of the intruder \( V \) or the force applied to the intruder \( F \) is specified. We then calculate the steady flow fields predicted by the nonlocal rheology using ABAQUS. The governing partial differential equations are the equilibrium equations \( \partial \sigma_{ij}/\partial x_j = 0 \), where inertia is neglected, since we are considering a quasistatic process and there is no gravitational body force since we are in two dimensions, and the differential relation for the granular fluidity (3). These are solved in conjunction with the constitutive equations (1) and (2) by means of a user element subroutine in ABAQUS. The mechanical boundary conditions are as described above, and, for the fluidity boundary conditions, we specify that \( n_i(\partial g/\partial x_i) = 0 \) at the inner and outer walls, where \( n_i \) is the outward surface normal. A detailed discussion of the intruder boundary conditions is given in Supplemental Material [27]. The necessary material parameters are \( \{\mu_s, b, A\} \). Following previous work involving glass beads [13,17,28], we take \( \mu_s = 0.3819 \) and \( b = 0.9377 \). For the two-dimensional problem, we take \( A = 1.8 \) (see Supplemental Material [27] for a justification of this selection).

In our first set of simulations, we determine the critical force \( F_c \). To this end, we fix the inner wall (\( \Omega = 0 \)) and specify normalized intruder velocities \( V\sqrt{\rho_s/P_a} \) spanning from \( 10^{-7} \) to \( 10^{-2} \). Each calculation quickly reaches a
The inner wall is then moved at a fixed speed in a tangential direction which opposes the force applied to the intruder (see Fig. 1). This sets up a primary flow characterized by an inner-wall-located shear band and induces a granular fluidity field which is nonzero everywhere. This fluidity field decays with radial distance from the inner wall and is quite small in the region of the intruder. However, fluidization of this region causes the intruder to begin creeping.

steady state, and we plot the normalized steady-state applied force $F/P_aD$ as a function of velocity in Fig. 2 (see Supplemental Material [27] for a demonstration of mesh insensitivity). This result was observed to be identical for $L/d = 18, 24, 34$. For normalized velocities greater than approximately $10^{-5}$, we see a clear rate dependence. In the calculations, this corresponds to a non-negligible portion of the domain achieving $\mu > \mu_s$, so that the $\mu\phi$ term in (3) has a substantial effect in this region. See the contour plot inset of the stress ratio $\mu$ in the region of the intruder; black regions denote where $\mu > \mu_s$.

The white region represents the intruder, which is moving upwards, and the region in which $\mu > \mu_s$ is denoted as black. Much of the region in front of and behind the intruder has met this condition. However, for $V\sqrt{\rho_s/P_a}$ less than $10^{-5}$, the force becomes nominally rate independent, plateauing to a constant value of $F_c/P_aD = 2.91$, which we denote as the critical force. In this case, the region around the intruder where $\mu > \mu_s$ is significantly smaller (see the inset in Fig. 2 for $V\sqrt{\rho_s/P_a} = 1 \times 10^{-7}$). The relation of Fig. 2 is consistent with analogous experimental observations [see Fig. 1(c) of Ref. [9] and Fig. 2(b) of Ref. [10]].

Next, we examine creep phenomenology when the inner wall is rotated. In our calculations, we first apply a force to the intruder less than $F_c$ and hold the force constant for a period of time. During this time, the intruder does not creep, and the granular fluidity is zero everywhere. The inner wall is then moved at a fixed speed in a tangential direction which opposes the force applied to the intruder (see Fig. 1). This sets up a primary flow characterized by an inner-wall-located shear band and induces a granular fluidity field which is nonzero everywhere. This fluidity field decays with radial distance from the inner wall and is quite small in the region of the intruder. However, fluidization of this region causes the intruder to begin creeping.

Steady state is quickly reached, and we denote the constant creep speed of the intruder as $V_c$. Figure 3 shows the normalized intruder creep speed $V_c\sqrt{\rho_s/P_a}$ as a function of the normalized inner wall speed $\Omega R_i \sqrt{\rho_s/P_a}$ for a fixed applied force $F/F_c = 0.75$ and intruder position $L/d = 24$, indicating a linear relationship with $V_c/\Omega R_i = 1.3 \times 10^{-3}$. This is a hallmark of rate independence and demonstrates that the material time scale $d\sqrt{\rho_s/P_a}$ is nominally irrelevant for $\Omega R_i \sqrt{\rho_s/P_a} \approx 10^{-3}$ and that the time scale dictating the creep speed of the intruder is that imposed by the inner wall speed. As the inner wall speed is increased further, a deviation from linearity is observed in the calculations, indicating an increased role of the local rheology. This rate-independent regime of creep was observed in all configurations of experiments [see Fig. 4(a) of Ref. [8], Fig. 2(b) of Ref. [9], and Fig. 3 of Ref. [10]].

The creep speed in the rate-independent regime depends on the force applied to the intruder $F/F_c$ as well as the distance between the intruder and the inner wall $L/d$. (We did not examine the effect of intruder size $D/d$ in this study.) The calculated dependence is shown as symbols in Fig. 4 for $L/d = 18, 24, 34$ and $\Omega R_i \sqrt{\rho_s/P_a} = 1 \times 10^{-5}$. For $F < F_c$, we observe that (i) the intruder...
creeps faster for higher applied force and (ii) the intruder creeps faster when it is closer to the inner wall for all applied forces—both intuitive notions. What is nonintuitive is that the relation between the creep speed and applied force for \( F < F_c \) and fixed \( L/d \) is nonlinear. In fact, the relation in this range is exponential in character, consistent with experiments (see Fig. 4 of Ref. [9] and Fig. 4 of Ref. [10]). Also plotted in Fig. 4 as a dashed line is the drag coefficient versus intruder velocity relationship of Fig. 2. As the applied force is increased past \( F_c \), the simulated data for different \( L/d \) converge and asymptotically approach the rate-dependent part of this relationship, which is dominated by the local rheology rather than nonlocal effects. The convergence of the data around \( F = F_c \) is qualitatively consistent with experiments [9]. (For more discussion of this point and additional simulated creep data, see Supplemental Material [27].)

The faster-than-linear character of the results of Fig. 4 may be understood by using the following examination of the differential relation (3), while the specifically exponential shape observed is assessed in Supplemental Material [27]. The differential relation (3) has the structure of an inhomogeneous Helmholtz equation with \( g_{\text{loc}} \) playing the role of a “source” term. When subjected to the boundary condition \( n_i (\partial g / \partial x_i) = 0 \) at the walls, the fluidity field will be zero everywhere if \( g_{\text{loc}} = 0 \) (or \( \mu < \mu_s \)) everywhere and will be nonzero everywhere if \( g_{\text{loc}} > 0 \) (or \( \mu > \mu_s \)) anywhere. In the present simulations, rotation of the inner wall sets up a small region at the inner wall where \( g_{\text{loc}} > 0 \)—a shear band—and, as one moves away from the shear band, the fluidity undergoes an exponential-like decay but remains nonzero. This fluidity field may equivalently be thought of as a radially dependent viscosity field with the viscosity lower closer to the shear band. This justifies the increase of creep speed with decreasing \( L/d \). A simplistic Stokes-flow view of the creep problem would indicate that the creep speed and applied force should be linearly related. The fact that this is not observed indicates that the presence of the intruder has an important influence on the fluidity field. This effect is summarized in Fig. 5. Figure 5(a) shows contour plots of the stress ratio field \( \mu \) in the region of the intruder (which is white and moving upwards) for two values of the applied force \( F/F_c \), 0.75 and 0.95, \( L/d = 24 \), and \( \Omega R_i \sqrt{\rho_s / \rho} = 1 \times 10^{-5} \). In both cases, the stress ratio \( \mu \) reaches and just exceeds the critical value \( \mu_s \), leading to \( g_{\text{loc}} > 0 \) in a very small region of the intruder boundary. In a sense, this “activates” an additional source at the intruder. As can be seen in Fig. 5(a), while this region is indiscernible for \( F/F_c = 0.75 \), more of the region surrounding the intruder reaches \( \mu_s \) for the case of \( F/F_c = 0.95 \), and, as a consequence, the “source” is stronger and the fluidity is increased in the region of the intruder. Figure 5(b) shows the fluidity \( g \) along a radial path from the inner wall to the intruder for \( F/F_c = 0.75 \) and 0.95. Near the inner wall, away from the intruder, the fluidity field is essentially the same in both cases. However, in the region of the intruder, the stronger source in the higher force case leads to a local fluidity which is approximately 6 times greater—or, equivalently, a viscosity which is 6 times less. This decrease in viscosity combined with the greater applied force leads to the faster-than-linear increase in creep speed with force observed in Fig. 4.

In conclusion, we have demonstrated that the nonlocal granular rheology is capable of describing the phenomenology of secondary flow fields [17], offering a means for more thorough, quantitative interpretation of secondary rheology experiments and providing a complete description of all relevant fields, including those that might be difficult to measure experimentally, such as stress. Finally, the fact that the model describes both primary flow fields [17] as well as secondary flow demonstrates the generality of the model and paves the way for modeling other complex phenomena in dense granular flow, such as flow anisotropy [10] and nonmonotonic rheology [29,30].

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