Stable Periodic Solutions to Lambda-Omega Lattice Dynamical Systems

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Abstract

In this manuscript we consider the stability of periodic solutions to Lambda-Omega lattice dynamical systems. In particular, we show that an appropriate ansatz casts the lattice dynamical system as an infinite-dimensional fast-slow differential equation. In a neighbourhood of the periodic solution an invariant slow manifold is proven to exist, and that this slow manifold is uniformly exponentially attracting. The dynamics of solutions on the slow manifold become significantly more complicated and necessitate recent results providing algebraic decay rates of synchronous solutions to systems of coupled oscillators. Of particular interest to our work in this manuscript is the stability of a rotating wave solution, recently found to exist in the Lambda-Omega systems studied herein.

1 Introduction

At their most general a lattice dynamical system (LDS) takes the form

\[ \dot{u}_\xi = g_\xi(\{u_\zeta\}_\zeta \in \Lambda), \quad \xi \in \Lambda. \]  

(1.1)

Here \( \Lambda \) is a discrete subset of \( \mathbb{R}^n \), commonly referred to as a lattice, and the \( u_\xi = u_\xi(t) \) are time-dependent functions indexed by the lattice. We consider the variables \( u_\xi \) for each \( \xi \in \Lambda \) to be coordinates in the state vector \( u = \{u_\xi\}_{\xi \in \Lambda} \) and \( g_\xi \) a function on these coordinates which governs the flow of each \( u_\xi \). Lattice models of the form (1.1) have been shown to be well-suited in such areas as chemical reaction theory \([15, 29]\), quantum mechanics \([27, 28, 33]\), models of neural networks \([10, 12]\), optics \([16]\) and material science \([5, 9]\). In many of these applications one considers the elements \( u_\xi \) to be interacting particles whose behaviour influences, and is influenced by, a finite subset of elements of the state vector \( u = \{u_\xi\}_{\xi \in \Lambda} \). Despite spatially continuous models such as partial differential equations (PDEs) being the traditional means by which such situations are characterized mathematically, it appears that the discrete nature of the lattice model is better suited to reflect the discrete nature of the physical setting which the models are attempting to describe.

In many interesting mathematical investigations of LDSs one typically considers the index set \( \Lambda \) to be countably infinite, thus making (1.1) an infinite system of coupled ordinary differential equations.
This infinite-dimensional setting of course necessitates more abstract and technical analytical tools to investigate the behaviour of solutions to systems of the form (1.1). In particular, the problem of determining local asymptotic stability of a known solution to (1.1) greatly increases in complexity when moving from the finite-dimensional to the infinite-dimensional setting. In the case when $\Lambda = \mathbb{Z}$ some authors have circumvented this difficulty through the use of comparison principles to obtain stability of traveling waves solutions to LDSs $[6, 18, 26, 30, 34, 35]$. The problem with this method is that it requires a number of assumptions on the model and can only capture a specific class of initial conditions that converge back to the given solution. This was exactly the point made in $[23]$ where the authors examine the local asymptotic stability of traveling waves with $\Lambda = \mathbb{Z}^2$ in a more traditional dynamical systems context.

In this work we aim to continue the discussion of local asymptotic stability of solutions to LDSs on higher-dimensional lattices by investigating the stability of periodic solutions to the so-called Lambda-Omega system

$$\dot{z}_{i,j} = \alpha(z_{i+1,j} + z_{i-1,j} + z_{i,j+1} + z_{i,j-1} - 4z_{i,j}) + z_{i,j}[\lambda(|z_{i,j}|) + i\omega(|z_{i,j}|, \alpha)], \quad (i,j) \in \mathbb{Z}^2,$$

(1.2)

which we write in terms of the time-dependent complex variables $z_{i,j} = z_{i,j}(t)$. Here $\alpha \geq 0$ is often referred to as the coupling coefficient and it describes the strength of interaction between neighbouring elements of the lattice. The inclusion of the terms $z_{i\pm 1,j}$ and $z_{i,j\pm 1}$ on the right-hand side of (1.2) represents a local coupling over the two-dimensional lattice $\mathbb{Z}^2$, referred to as nearest-neighbour coupling. Such a coupling can be derived as a leading order approximation of the typical five point discretization of the Laplacian differential operator, leading one to view system (1.2) as a spatially discretized reaction-diffusion PDE (see $[2, \text{Section 1.1}]$ for full details). When $\alpha \geq 0$ is taken to be a small parameter in the system, one may alternatively view (1.2) as an infinite system of weakly coupled oscillators, since when $\alpha = 0$ the $z_{i,j}$ act independently each other. The specifics of the functions $\lambda$ and $\omega$ will be detailed later in this manuscript, but the important characteristic to consider at this point is that they only depend on the modulus of the complex variable $z_{i,j}$, endowing (1.2) with a rotating symmetry invariance. In this way, the choices for $\lambda$ and $\omega$ are meant to mimic the behaviour of the normal form of a Hopf bifurcation in the ordinary differential equation setting.

Since their introduction by Howard and Kopell, Lambda-Omega PDEs have long been studied as an archetype for oscillatory behaviour in reaction-diffusion systems $[24]$. In particular, Lambda-Omega PDEs are presented as generalizations of the complex Ginzburg-Landau equation, which is well-known to be the truncated normal form of a reaction-diffusion equation undergoing a Hopf bifurcation $[8]$. To date Ginzburg-Landau PDEs have been shown to manifest themselves as the dominant leading order perturbation in a wide class of partial differential equations, including for example the Swift-Hohenberg equation, therefore testifying to the universality of these equations $[31]$. Hence, Lambda-Omega systems should not be viewed as a specific model of a physical phenomenon, but as a paradigm for understanding periodic behaviour.

Many investigations have demonstrated that Lambda-Omega PDEs exhibit rotating wave solutions $[8, 14, 17, 24]$, thus prompting the recent investigation into rotating wave solutions to LDSs of the type (1.2) $[1, 2]$. Hence, this work provides a natural follow-up investigation to $[1, 2]$ by providing a series of sufficient conditions which can determine the stability of rotating wave solutions to (1.2). We dedicate an entire section of this manuscript to discussing how our assumptions apply to rotating wave solutions to (1.2), thus providing a nuanced discussion of the stability of rotating waves while also providing new methods for determining local asymptotic stability of solutions to LDSs. The importance of our focus on the stability of rotating wave solutions to (1.2) is that even in the PDE setting little is known about the
stability of rotating waves. Therefore any insight into the stability of rotating waves should be of value to a wide variety of researchers due to the prevalence of rotating waves in chemical and biological systems. Numerical evidence and heuristic arguments in [20] appear to indicate that single-armed rotating waves should be locally asymptotically stable in the reaction-diffusion PDE setting, but this remains to be verified rigorously. Here we add to this mounting evidence by providing numerical simulations that indicate that rotating wave solutions should in fact satisfy the sufficient conditions for local asymptotic stability laid out in this manuscript.

In this work we show that after introducing an appropriate ansatz into system (1.2) it can be interpreted as an infinite-dimensional fast-slow dynamical system when \( 0 \leq \alpha \ll 1 \). The fast-slow nature of the resulting dynamical system requires the understanding of the asymptotic behaviour of solutions on two different time-scales: one fast and one slow. We show that an invariant manifold in the form of an infinite-dimensional torus persists for sufficiently small \( \alpha \geq 0 \), and that this invariant manifold is locally asymptotically stable with an exponential rate of decay. Solutions on the invariant manifold evolve in the slow time variable, leading to the nomenclature that the invariant manifold is a slow manifold. Upon reducing to the slow manifold we are able to extend previous methods presented in [3] to investigate the stability of solutions on the manifold. We find that on the slow manifold solutions decay back to equilibrium at an algebraic (as opposed to exponential) rate provided they start sufficiently close to that equilibrium. The notion of closeness is one that requires extra attention in this infinite-dimensional setting since it depends on the Banach space in which one is measuring distance. Hence, we pay special attention to describing the Banach spaces in which initial conditions belong to, as well as how solutions decay with respect to a variety of different norms.

This paper is organized as follows. We begin with a discussion of the relevant Banach spaces which are used throughout, as well as present a series of results pertaining to a family of semi-norms which will be integral to this work. In Section 3 we detail the assumptions made on system (1.2) and provide the major results of this work. The entirety of Section 4 is dedicated to proving the existence and local asymptotic stability of the invariant slow manifold described above. Then, in Section 5 we prove the local asymptotic stability of solutions on the slow manifold. Section 6 provides a brief overview of local asymptotic stability in the spatially continuous heat equation PDE in an effort to lead to a greater understanding of the spatially discrete setting of LDSs studied herein. We then turn our attention to rotating waves in Section 7 where we systematically show that at least three of the four assumptions made must hold for these solutions, as well as provide numerical evidence and a heuristic argument that the fourth should hold as well. Finally, this paper concludes with Section 8 which provides concluding remarks on the work undertaken in this paper, as well as details avenues for future work.

2 Spatial Settings

Prior to presenting the main hypotheses and results of this work, we provide the following discussion regarding the appropriate spatial settings for solutions of (3.1). To begin, the Banach spaces which will be of primary interest throughout this work will be sequence spaces indexed by a countably infinite index set \( \mathbb{Z}^2 \). In particular, our attention will be focussed on the spaces \( \ell^p(\mathbb{Z}^2) \) with \( p \in [1, \infty] \). They are defined as follows:

\[
\ell^p(\mathbb{Z}^2) = \left\{ x = \{x_{i,j}\} \in \mathbb{Z}^2 : \sum_{(i,j) \in \mathbb{Z}^2} |x_{i,j}|^p < \infty \right\},
\]

(2.1)
for all $p \in [1, \infty)$ and

$$\ell^\infty(\mathbb{Z}^2) = \left\{ x = \{x_{i,j}\}_{(i,j)\in\mathbb{Z}^2} : \sup_{n\in\mathbb{Z}^2} |x_{i,j}| < \infty \right\}. \quad (2.2)$$

It is well-known that $\ell^p(\mathbb{Z}^2)$ is complete (and therefore a Banach space) under the norm

$$\|x\|_p = \left( \sum_{(i,j)\in\mathbb{Z}^2} |x_{i,j}|^p \right)^{\frac{1}{p}}, \quad (2.3)$$

and similarly the norm associated to $\ell^\infty(\mathbb{Z}^2)$ is given by

$$\|x\|_\infty = \sup_{(i,j)\in\mathbb{Z}^2} |x_{i,j}|. \quad (2.4)$$

Since the index set will always be $\mathbb{Z}^2$, for the ease of notation we will write $\ell^p(\mathbb{Z}^2)$ as $\ell^p$. It should be noted that $\ell^1 \subset \ell^p$ for every $p > 1$, and hence posing solutions to (3.1) in $\ell^1$ will allow for the discussion of the behaviour of solutions with respect to all $\ell^p$ norms. Due to the subscript indices of elements in $\ell^p$, throughout this manuscript we will write initial conditions as $x_0 = \{x_{0,i,j}\}_{(i,j)\in\mathbb{Z}^2}$. This avoids the confusion created by using the traditional notation of $x_0$, where the meaning of the subscript could be ambiguous to the reader since $x_0$ could represent an element of $\ell^p$ or a single element of the sequence of an element in $\ell^p$.

For convenience, throughout this manuscript we will introduce the shorthand

$$\sum_{i',j'}(x_{i',j'} - x_{i,j}) := (x_{i+1,j} - x_{i,j}) + (x_{i-1,j} - x_{i,j}) + (x_{i,j+1} - x_{i,j}) + (x_{i,j-1} - x_{i,j}). \quad (2.5)$$

Then, using this shorthand we will consider the real-valued functions on $\ell^p$ given as

$$Q_p(x) := \left( \sum_{(i,j)\in\mathbb{Z}^2} \sum_{(i',j')\in\mathbb{Z}^2} |x_{i',j'} - x_{i,j}|^p \right)^{\frac{1}{p}}, \quad (2.6)$$

for all $x \in \ell^p$, $p \in [1, \infty)$. Similarly, define $Q_\infty$ as

$$Q_\infty(x) := \sup_{(i,j)\in\mathbb{Z}^2} \sum_{(i',j')\in\mathbb{Z}^2} |x_{i',j'} - x_{i,j}|,$$

for all $x \in \ell^\infty$. The terms $(x_{i',j'} - x_{i,j})$ can be interpreted as discrete directional derivatives along the horizontal and vertical directions of the lattice. Hence, the functions $Q_p$ can be understood as the discrete analogue of the norm of the gradient of a function in the Lesbesgue measure spaces. We now provide the following lemma to show that the functions $Q_p$ are indeed well-defined.

**Lemma 2.1.** For every $1 \leq p \leq p' \leq \infty$ and $x \in \ell^p$ we have:

$$Q_{p'}(x) \leq Q_p(x) \leq 8\|x\|_p.$$

**Proof.** The proof of the left inequality follows in exactly the same way as showing that $\|x\|_{p'} \leq \|x\|_p$, and therefore we are left to prove the rightmost inequality. Begin by noting that a simple application of the triangle inequality gives

$$|x_{i',j'} - x_{i,j}| \leq |x_{i',j'}| + |x_{i,j}|,$$
for every \((i, j) \in \mathbb{Z}^2\) and a nearest-neighbour \((i', j')\). Then, for \(1 \leq p < \infty\), the triangle inequality on the \(\ell^p\) spaces imply that

\[
Q_p(x) \leq \left( \sum_{(i, j) \in \mathbb{Z}^2} |x_{i+1,j}|^p \right)^{\frac{1}{p}} + \left( \sum_{(i, j) \in \mathbb{Z}^2} |x_{i-1,j}|^p \right)^{\frac{1}{p}} + \left( \sum_{(i, j) \in \mathbb{Z}^2} |x_{i,j+1}|^p \right)^{\frac{1}{p}} + \left( \sum_{(i, j) \in \mathbb{Z}^2} |x_{i,j-1}|^p \right)^{\frac{1}{p}} + 4 \left( \sum_{(i, j) \in \mathbb{Z}^2} |x_{i,j}|^p \right)^{\frac{1}{p}} = 8 \|x\|_p,
\]

as claimed. The case when \(p = \infty\) follows in a similar way, and is omitted.

Throughout this work we will make use of the following simple bound.

**Lemma 2.2.** For every \(p \in [1, \infty)\) and \(x \in \ell^p\) we have

\[
\left( \sum_{(i, j) \in \mathbb{Z}^2} \left( \sum_{(i', j') \in \mathbb{Z}^2} |x_{i', j'} - x_{i,j}| \right)^p \right)^{\frac{1}{p}} \leq 4Q_p(x).
\]

**Proof.** Begin by fixing some \(p \in [1, \infty)\) and let \(q \in [1, \infty)\) be its Hölder conjugate. Then, for all \(x \in \ell^p\) Hölder’s inequality gives

\[
\left( \sum_{(i, j) \in \mathbb{Z}^2} \left( \sum_{(i', j') \in \mathbb{Z}^2} |x_{i', j'} - x_{i,j}| \right)^p \right)^{\frac{1}{p}} \leq 4 \left( \sum_{(i', j') \in \mathbb{Z}^2} |x_{i', j'} - x_{i,j}|^p \right)^{\frac{1}{p}},
\]

uniformly in \((i, j) \in \mathbb{Z}^2\), since \(q \geq 1\). The stated inequality now follows since

\[
\left( \sum_{(i, j) \in \mathbb{Z}^2} \left( \sum_{(i', j') \in \mathbb{Z}^2} |x_{i', j'} - x_{i,j}| \right)^p \right)^{\frac{1}{p}} \leq 4 \left( \sum_{(i, j) \in \mathbb{Z}^2} \sum_{(i', j') \in \mathbb{Z}^2} |x_{i', j'} - x_{i,j}|^p \right)^{\frac{1}{p}} = 4Q_p(x).
\]

We note that the functions \(Q_p\) should be interpreted as semi-norms since they annihilate constant sequences\(^\dagger\). However, since the constant sequences only belong to \(\ell^\infty\), this will not pose a problem to our analysis since we primarily focus on elements in \(\ell^1\), which does not include the constant sequences. Furthermore, we will see that it is indeed an understanding of the behaviour of solutions with respect to the \(Q_p\) semi-norms that influences our understanding of the behaviour of solutions with respect to the \(\ell^p\) norms. Hence, this work aims to convince the reader that a complete discussion of stability in \([3.1]\) requires the introduction of the \(Q_p\) semi-norms. Having now done so, we are now able to present the hypotheses and main results of this work.

\(^\dagger\)In fact, one can easily prove that \(Q_p(x) = 0\) if, and only if, \(x\) is a constant sequence.
3 Hypotheses and Main Results

To begin, using the shorthand (2.5) we find that (1.2) can be written compactly as

\[
\dot{z}_{i,j} = \alpha \sum_{i',j'} (z_{i',j'} - z_{i,j}) + z_{i,j} [\lambda (|z_{i,j}|) + i\omega (|z_{i,j}|, \alpha)], \quad (i, j) \in \mathbb{Z}^2.
\]  

(3.1)

We make the following hypothesis on the functions \(\lambda\) and \(\omega\) in the differential equation (3.1).

**Hypothesis 1.** The functions \(\lambda\) and \(\omega\) in (3.1) satisfy the following:

1. \(\lambda : [0, \infty) \to \mathbb{R}\) is continuously differentiable and there exists some \(a > 0\), with the property that \(\lambda(a) = 0\) and \(\lambda'(a) < 0\).
2. \(\omega = \omega(R, \alpha)\) is of the form

\[
\omega(R, \alpha) = \omega_0(\alpha) + \alpha \omega_1(R, \alpha),
\]  

(3.2)

for some function \(\omega_0 : \mathbb{R} \to \mathbb{R}\) and \(\omega_1(R, \alpha) : [0, \infty) \times \mathbb{R} \to \mathbb{R}\) twice continuously differentiable with \(\omega_1(a, \alpha) = 0\) and \(\partial_1 \omega_1(a, \alpha) = 0\) for all \(\alpha \in \mathbb{R}\).

These conditions are based upon the assumptions laid out in the first demonstration of the existence of rotating waves in the continuous spatial setting [8, 17]. Furthermore, conditions of this form were used in [2] to demonstrate the existence of rotating wave solutions to equation (3.1), which serve as a major motivation for the analysis herein. We note that the assumptions to demonstrate the existence of rotating wave solutions to (3.1) only required that \(\lambda'(a) \neq 0\), but now to demonstrate asymptotic stability of solutions to (3.1) we require \(\lambda'(a) < 0\). To understand why this is the case, begin by setting \(\alpha = 0\) in (3.1) to arrive at the infinite system of uncoupled equations

\[
\dot{z}_{i,j} = z_{i,j} [\lambda (|z_{i,j}|) + i\omega_0(|z_{i,j}|, 0)],
\]

for all \((i, j) \in \mathbb{Z}^2\). Decomposing each \(z_{i,j}\) into polar variables using the ansatz

\[
z_{i,j}(t) = r_{i,j}(t)e^{i\theta_{i,j}(t)}
\]  

(3.3)

results in the set of ordinary differential equations

\[
\dot{r}_{i,j} = r_{i,j} \lambda(r_{i,j}),
\]

\[
\dot{\theta}_{i,j} = \omega_0(0),
\]  

(3.4)

for each \((i, j) \in \mathbb{Z}^2\). Taking \(r_{i,j} = a\) leads to a periodic solution of the form

\[
z_{i,j}(t) = ae^{i(\omega_0(0)t + \theta_0(j))},
\]  

(3.5)

where \(\theta_0(j) \in S^1\) is an initial phase value for each \((i, j) \in \mathbb{Z}^2\). Hence, assuming \(\lambda'(a) \neq 0\), each periodic solution of the form (3.5) to (3.4) falls into one of two categories: locally attracting when \(\lambda'(a) < 0\) and locally repelling when \(\lambda'(a) > 0\). Since we are interested in the stability of solutions to (3.1) with \(\alpha\) in a connected neighbourhood to the right of \(\alpha = 0\), we therefore must focus on the case \(\lambda'(a) < 0\).

Turning now to the function \(\omega\), we first note that many applications simply work with \(\omega\) as a constant function (or at least independent of its first argument). When extending to non-constant
\( \omega \) functions, similar work exploring rotating waves in Lambda-Omega systems on finite lattices has considered functions \( \omega \) to be slight perturbations off of a constant function \[^{[13]}\]. We see that indeed this is the case when \( \alpha \geq 0 \) is taken to be a small parameter in the system, which therefore measures the deviation of \( \omega \) from a constant function. Furthermore, the condition \( \omega_1(a, \alpha) = 0 \) and \( \partial_1 \omega_1(a, \alpha) = 0 \) implies that

\[
\omega_1(R, \alpha) = \mathcal{O}(|R - a|^2),
\]

simply using Taylor’s Theorem. This condition is very similar to that which was assumed by Cohen et. al. in their proof of existence of spiral waves in the spatially continuous reaction-diffusion setting \[^{[8]}\]. Particularly, they assumed the Hölder regularity condition

\[
\omega_1(R, \alpha) = \mathcal{O}(|R - a|^{1+\mu}),
\]

for some \( \mu > 0 \). It would be natural to conjecture that the results of this manuscript would remain true for any \( \mu > 0 \), albeit with some minor technical hurdles to overcome, but for the convenience of presenting the results here we simply take \( \mu = 1 \).

To analyze the full system \(^{[3.1]}\) we follow in a similar way in which we inspected the uncoupled system \(^{[3.4]}\) above and introduce the polar decomposition

\[
z_{i,j} = r_{i,j}e^{i(\omega_1(t)+\theta_{i,j})},
\]

with \( r_{i,j} = r_{i,j}(t) \) and \( \theta_{i,j} = \theta_{i,j}(t) \) for each \( (i,j) \in \mathbb{Z}^2 \). Then the LDS \(^{[3.1]}\) can now be written in polar form as

\[
\begin{align*}
\dot{r}_{i,j} &= \alpha \sum_{i',j'} \left( r_{i',j'} \cos(\theta_{i',j'} - \theta_{i,j}) - r_{i,j} \right) + r_{i,j} \lambda (r_{i,j}), \\
\dot{\theta}_{i,j} &= \alpha \sum_{i',j'} \frac{r_{i',j'}}{r_{i,j}} \sin(\theta_{i',j'} - \theta_{i,j}) + \alpha \omega_1(r_{i,j}, \alpha), \quad (i,j) \in \mathbb{Z}^2.
\end{align*}
\]

Note that for \( 0 \leq \alpha \ll 1 \) the phase components become singularly perturbed, hence giving that in the small \( \alpha > 0 \) limit this polar decomposition is of the form of a fast-slow system of ordinary differential equations. We now make the following assumption.

**Hypothesis 2.** There exists \( \alpha^* > 0 \) such that for all \( \alpha \in [0, \alpha^*] \) there exists a steady-state solution, denoted \( \{ \bar{r}_{i,j}(\alpha), \bar{\theta}_{i,j}(\alpha) \}_{(i,j) \in \mathbb{Z}^2} \), to \(^{[3.7]}\). That is,

\[
0 = \alpha \sum_{i',j'} \left( \bar{r}_{i',j'}(\alpha) \cos(\theta_{i',j'} - \theta_{i,j}) - \bar{r}_{i,j}(\alpha) \right) + \bar{r}_{i,j}(\alpha) \lambda (\bar{r}_{i,j}(\alpha)),
\]

\[
0 = \sum_{i',j'} \frac{\bar{r}_{i',j'}(\alpha)}{\bar{r}_{i,j}(\alpha)} \sin(\bar{\theta}_{i',j'}(\alpha) - \bar{\theta}_{i,j}(\alpha)) + \omega_1(\bar{r}_{i,j}(\alpha), \alpha),
\]

for all \( \alpha \in [0, \alpha^*] \). Furthermore, \( \bar{r}_{i,j}(0) = a \) and there exists a constant \( C_r > 0 \) such that

\[
|\bar{r}_{i,j}(\alpha) - a| \leq C_r \alpha
\]

**Remark 1.** Without loss of generality we will restrict \( \alpha^* > 0 \) such that \( |\bar{r}_{i,j}(\alpha) - a| \leq \frac{\alpha}{2} \) for all \( (i,j) \in \mathbb{Z}^2 \) using \(^{[3.8]}\). This will allow our analysis to be bounded away from the singularity at \( \bar{r}_{i,j}(\alpha) = 0 \) in the differential equation for \( \bar{\theta}_{i,j} \) in \(^{[3.7]}\).
One sees that Hypothesis 2 results in a periodic solution \( \{z_{i,j}(t; \alpha)\}_{(i,j) \in \mathbb{Z}^2} \) of the form
\[
z_{i,j}(t; \alpha) = \bar{r}_{i,j}(\alpha)e^{i(\omega_0(\alpha)t + \bar{\theta}_{i,j}(\alpha))},
\]
for all \((i, j) \in \mathbb{Z}^2\). Here we see that each element of the lattice is oscillating with a frequency of \(2\pi/\omega_0(\alpha)\), but potentially differs through the amplitude of its oscillation, \(\bar{r}_{i,j}(\alpha)\), and/or a phase-lag, \(\bar{\theta}_{i,j}(\alpha)\). Of course there is a trivial choice for a solution satisfying Hypothesis 2 given by \(\bar{r}_{i,j}(\alpha) = a\) and \(\bar{\theta}_{i,j}(\alpha) = 0\) for all \((i, j) \in \mathbb{Z}^2\) and \(\alpha \geq 0\). Aside from this trivial solution, it was shown in [2] that there also exists a rotating wave solution satisfying Hypothesis 2 for sufficiently small \(\alpha \geq 0\). On top of these two solutions, the methods employed to obtain the rotating wave solution can be easily extended to obtain various other steady-states of (3.7) which satisfy Hypothesis 2 leading to oscillatory solutions of (3.1).

To properly state our results we centre system (3.7) at the steady-state \((\bar{r}(\alpha), \bar{\theta}(\alpha))\). Let us introduce the changes of variable given by
\[
\begin{align*}
\bar{r}_{i,j} &= \bar{r}_{i,j}(\alpha) + s_{i,j}, \\
\bar{\theta}_{i,j} &= \bar{\theta}_{i,j}(\alpha) + \psi_{i,j},
\end{align*}
\]
for all \((i, j) \in \mathbb{Z}^2\). Letting \(s = \{s_{i,j}\}_{(i,j) \in \mathbb{Z}^2}\) and \(\psi = \{\psi_{i,j}\}_{(i,j) \in \mathbb{Z}^2}\), we define the function
\[
F(s, \psi, \alpha) = \{F_{i,j}(s, \psi, \alpha)\}_{(i,j) \in \mathbb{Z}^2}
\]
to describe the radial part as
\[
\begin{align*}
F_{i,j}(s, \psi, \alpha) &= \alpha \sum_{i', j'}\left[\bar{r}_{i',j'}(\alpha) + s_{i',j'}\cos(\bar{\theta}_{i',j'}(\alpha) + \psi_{i',j'} - \bar{\theta}_{i,j}(\alpha) - \psi_{i,j}) - (\bar{r}_{i,j}(\alpha) + s_{i,j})\right] \\
&\quad + (\bar{r}_{i,j}(\alpha) + s_{i,j})\lambda(\bar{r}_{i,j}(\alpha) + s_{i,j}),
\end{align*}
\]
so that we have \(F(0, 0, \alpha) = 0\) for all \(\alpha \in [0, \alpha^*]\). Furthermore, we define the function
\[
G(s, \psi, \alpha) = \{G_{i,j}(s, \psi, \alpha)\}_{(i,j) \in \mathbb{Z}^2}
\]
to describe the phase components as
\[
\begin{align*}
G_{i,j}(s, \psi, \alpha) &= \sum_{i', j'}\left[\left(\frac{\bar{r}_{i',j'}(\alpha)}{\bar{r}_{i,j}(\alpha)} + s_{i',j'}\right)\sin(\bar{\theta}_{i',j'} + \psi_{i',j'} - \bar{\theta}_{i,j} - \psi_{i,j})\right] + \omega_1(\bar{r}_{i,j}(\alpha) + s_{i,j}, \alpha),
\end{align*}
\]
so that \(G(0, 0, \alpha) = 0\) for all \(\alpha \in [0, \alpha^*]\). This therefore leads to the system which we will focus on throughout this work
\[
\begin{align*}
\dot{s} &= F(s, \psi, \alpha), \\
\dot{\psi} &= \alpha G(s, \psi, \alpha).
\end{align*}
\]

Written in the form (3.11) it becomes easier to identify that when \(0 < \alpha \ll 1\) our system resembles a fast-slow dynamical system, with the major caveat that we are working in infinite dimensions. Our approach to describing the behaviour of solutions to (3.11) will be motivated by the study of finite-dimensional fast-slow dynamical systems, but now the infinite-dimensionality of the problem requires one to be more careful since the notion of distance is dependent on the underlying infinite dimensional phase space in which the solutions exist in. The first major result of this work describes the persistence of an infinite-dimensional invariant manifold in (3.11), for which the proof is left to Section 4.
**Theorem 3.1.** Assume Hypothesis 1 and 2. Then, there exists \(\alpha_0 \in (0, \alpha^*]\) such that the following is true: there exists a function
\[
\sigma : \ell^1 \times [0, \alpha_0] \to \ell^1
\]
such that the set
\[
\{(s, \psi) : s = \sigma(\psi, \alpha), \psi \in \ell^1\}
\]
is an invariant manifold of system (3.11) for all \(\alpha \in [0, \alpha_0]\). The function \(\sigma\) satisfies the following properties:
\[
\sigma(0, \alpha) = 0, \\
\sup_{\psi \in \ell^1} \|\sigma(\psi, \alpha)\|_\infty \leq \sqrt{\alpha}, \\
\|\sigma(\psi, \alpha) - \sigma(\tilde{\psi}, \alpha)\|_p \leq \sqrt{\alpha}Q_p(\psi - \tilde{\psi}),
\]
for all \(p \in [1, \infty]\), \(\psi, \tilde{\psi} \in \ell^1\), and \(\alpha \in [0, \alpha_0]\). Furthermore, this invariant manifold is asymptotically exponentially stable. That is, there exists \(\delta^*, \beta > 0\) such that for all \(\delta \in (0, \delta^*]\), if
\[
\|s(0) - \sigma(\psi(0), \alpha)\|_1 \leq \delta,
\]
then
\[
\|s(t) - \sigma(\psi(t), \alpha)\|_1 \leq 2\delta e^{-\beta t}
\]
for all \(t \geq 0\) and \(\alpha \in [0, \alpha_0]\).

Theorem 3.1 allows one to reduce the dynamics of (3.11) to the invariant manifold to understand the behaviour of the phase components, \(\psi\). When put back into the single complex variable \(z_{i,j}\), this invariant manifold represents an infinite dimensional invariant torus given by
\[
\psi \mapsto \{(\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi, \alpha)) e^{i(\tilde{\theta}_{i,j}(\alpha) + \psi)}\}_{i,j} \in \mathbb{Z}^2,
\]
where we write \(\sigma(\psi) = \{\sigma_{i,j}(\psi)\}_{(i,j) \in \mathbb{Z}^2}\). Then, to extend Theorem 3.1 by examining the stability of solutions on the invariant manifold, we must first define the linear operator \(L_\alpha\) acting on the sequences \(x = \{x_{i,j}\}_{(i,j) \in \mathbb{Z}^2}\) by
\[
[L_\alpha x]_{i,j} = \sum_{i' j'} \cos(\tilde{\theta}_{i',j'}(\alpha) - \tilde{\theta}_{i,j}(\alpha)) (x_{i',j'} - x_{i,j}),
\]
for all \((i, j) \in \mathbb{Z}^2\). From the fact that each index \((i, j) \in \mathbb{Z}^2\) has exactly four nearest-neighbours, it is a straightforward exercise to find that \(L_\alpha : \ell^p \to \ell^p\) is a bounded linear operator for all \(p \in [1, \infty]\) and \(\alpha \in [0, \alpha^*]\), with uniformly bounded operator norm. Moreover, one may use the methods of [1, Proposition 6.2] to show that \(L_\alpha : \ell^\infty \to \ell^\infty\) is not a Fredholm operator, which implies that forward time exponential dichotomies of the \(\ell^\infty\) norm of the solutions to the linear ordinary differential equation
\[
\dot{x} = L_\alpha x
\]
cannot be obtained for arbitrary initial conditions \(x^0 \in \ell^\infty\). When every \(\cos(\tilde{\theta}_{i',j'}(\alpha) - \tilde{\theta}_{i,j}(\alpha))\) term is nonnegative the result [4, Theorem 2.1] then extends this result to give that the spectrum of \(L_\alpha : \ell^p \to \ell^p\) for every \(p \in [1, \infty]\) must therefore have nontrivial intersection with the imaginary axis of the complex plane, implying that forward time exponential dichotomies of the solutions to (3.14) cannot be obtained.
for initial conditions in any of the $\ell^p$ sequence spaces. The following hypothesis gives that although exponential dichotomies cannot be obtained, we still assume that solutions to (3.14) exhibit an algebraic decay in the $\ell^p$ norms.

**Hypothesis 3.** Let $L_\alpha$ act on the sequences indexed by $\mathbb{Z}^2$ as in (3.13). There exists constants $\eta, C_L > 0$ such that for all $x^0 \in \ell^1$, $\alpha \in [0, \alpha^*]$ and $t \geq 0$ we have

\[
\|e^{L_\alpha t}x^0\|_p \leq C_L(1 + t)^{-1 + \frac{1}{p}}\|x^0\|_1,
\]

\[
Q_p(e^{L_\alpha t}x^0) \leq C_L(1 + t)^{-1 + \frac{1}{p} - \eta}\|x^0\|_1,
\]

where $e^{L_\alpha t}$ is the semi-group with infinitesimal generator given by $L_\alpha$.

The decay rates stated in Hypothesis 3 are not arbitrary, but come from the work of [3]. That is, it was shown in [3] that when $\cos(\bar{\theta}_i,j^{*}(\alpha) - \bar{\theta}_i,j^{*}(\alpha)) \geq 0$ for all $(i, j) \in \mathbb{Z}^2$, the linear operator (3.13) can be interpreted as a graph Laplacian operator associated to an infinite graph with vertex set lying in one-to-one correspondence with the indices of the lattice. Then, it is shown that the geometry of this underlying graph can be used to obtain the decay rates (3.15). We refrain from going into full detail here since it would require the definition of a multitude of graph-related terminology which is only applicable to the statement of Hypothesis 3. We simply note that the reader can easily verify when a solution $\bar{\theta}(\alpha)$ leads to these decay rates by following the work in [3, Section 6]. Furthermore, the value $\eta > 0$ comes from [19, Theorem 2.32], which is a study of random walks on infinite weighted graphs, where, unfortunately, there is little indication of what exactly $\eta > 0$ should be since the work where it comes from was conducted in complete generality. In Section 6 we provide a heuristic argument that we should in fact have $\eta = 1$ in our present scenario, although this remains to be verified rigorously.

**Hypothesis 4.** There exists a constant $C_{\text{sol}} > 0$ and $p^* \geq 1$ such that for all $\alpha \in [0, \alpha^*]$ we have

\[
\sum_{(i, j) \in \mathbb{Z}^2} |\bar{r}_{i,j}(\alpha) - a|^{p^*} \leq C_{\text{sol}},
\]

\[
\sum_{(i, j) \in \mathbb{Z}^2} \sum_{i', j'} |\sin(\bar{\theta}_{i',j'}^{*}(\alpha) - \bar{\theta}_{i,j}(\alpha))|^{p^*} \leq C_{\text{sol}},
\]

and $\frac{1}{p^*} + \eta > 1$, where $\eta > 0$ is the constant in (3.15). We further denote $q^* \geq 1$ to be the Hölder conjugate of $p^*$ in that $\frac{1}{p^*} + \frac{1}{q^*} = 1$.

Hypothesis 3 is certainly the most technical assumption made and requires the most effort to verify. In some sense Hypothesis 4 states that the radial solutions are localized off the homogeneous state $r_{i,j} = a$, and that the phase components quickly become close in value to their nearest-neighbours as one moves out from the centre of the lattice. In Section 7 we will discuss these assumptions as they relate to the aforementioned rotating wave solution which satisfies Hypothesis 2. We also note that the condition $\frac{1}{p^*} + \eta > 1$ is very important to the investigation that follows. The following theorem provides the first instance in which it becomes relevant since we see that the nonlinear stability of the phase components does not fully mimic the linear decay rates stated in (3.15).

**Theorem 3.2.** Assume Hypotheses 1-4. Then there exists $\alpha_1 \in (0, \alpha^*]$ and a constant $\varepsilon^* > 0$ such that for all $\varepsilon \in [0, \varepsilon^*]$ and $s^0, \psi^0 \in \ell^1$ with the property that

\[
\|s^0 - \sigma(\psi^0, \alpha)\|_1 \leq \varepsilon, \quad \|\psi^0\|_1 \leq \varepsilon,
\]

(3.18)
there exists a unique solution of \((3.11)\) for all \(t \geq 0\) and \(\alpha \in [0, \alpha_1]\), denoted \((s(t), \psi(t))\), satisfying the following properties:

1. \(s(0) = s^0\) and \(\psi(0) = \psi^0\).
2. \((s(t), \psi(t)) \in \ell^1 \times \ell^1\) for all \(t \geq 0\).
3. There exists a \(\beta, C_\psi > 0\) such that
   \[
   \|s(t) - \sigma(\psi(t), \alpha)\|_1 \leq 2\varepsilon e^{-\beta t},
   \]
   and
   \[
   \|
   \psi(t)\|_p \leq \varepsilon C_\psi (1 + \alpha t)^{-1 + \eta},
   \]
   \[
   Q_p(\psi(t)) \leq \varepsilon C_\psi (1 + \alpha t)^{-\min\{1 - \frac{1}{p} + \eta, \frac{1}{p^*} + \eta\}},
   \]
   for all \(t \geq 0, \alpha \in [0, \alpha_1]\), and \(p \in [1, \infty]\).

**Remark 2.** Theorem 3.2 extends Theorem 3.1 by saying that if we choose an initial condition \((s^0, \psi^0)\) ∈ \(\ell^1 \times \ell^1\) sufficiently close to the steady-state \((s, \psi) = (0, 0)\) in the system \((3.11)\), we obtain exponential decay onto the invariant manifold, along with algebraic decay of the phase component \(\psi(t)\) with respect to the \(\ell^p\) norms. Moreover, if we write

\[
\rho(t) := s(t) - \sigma(\psi(t), \alpha),
\]

then \(\rho(t)\) describes the deviation from the invariant manifold which is decaying exponentially. From the properties of the function \(\sigma\), given in \((3.12)\), we have that

\[
\|
\sigma(\psi(t), \alpha)\|_p \leq \sqrt{\alpha} Q_p(\psi(t)) \leq \sqrt{\alpha} \varepsilon C_\psi (1 + \alpha t)^{-\min\{1 - \frac{1}{p} + \eta, \frac{1}{p^*} + \eta\}},
\]

which shows that the component of \(s(t)\) that belongs to the invariant manifold decays at a slightly faster rate than \(\psi(t)\) in each \(\ell^p\) norm. We further note the different timescales in which the decay is taking place, where all algebraic decay is with respect to the slow-time variable \(\alpha t\).

**Example.** As previously mentioned, there exists a trivial solution to \((3.7)\) given by \(\bar{r}_{i,j}(\alpha) = a\) and \(\bar{\theta}_{i,j}(\alpha) = 0\) for all \((i, j) \in \mathbb{Z}^2\) and \(\alpha \geq 0\). This solution leads to a synchronous periodic solution to \((3.1)\) of the form \(z_{i,j}(t) = ae^{i\omega_0 t}\) for all \((i, j) \in \mathbb{Z}^2\). It was already noted that this trivial solution satisfies Hypothesis 2 and Section 6.1 proves that Hypothesis 3 holds. Finally, Hypothesis 4 trivially holds for any \(C_{\text{sol}} > 0\) and \(p^* = 1\), giving that \(\frac{1}{p^*} + \eta > 1\) for any \(\eta > 0\). This therefore shows that Theorem 3.2 can be applied to this trivial solution to demonstrate the stability of such a synchronous periodic solution to \((3.1)\).

The proof of Theorem 3.2 that remains after proving Theorem 3.1 is left to Section 5. We first provide a pair of useful integral inequalities in Subsection 5.1 which are used throughout Section 5. Then, in Subsection 5.2, we study the decay properties of a semigroup closely related to the semi-group \(e^{L_\alpha t}\) given in Hypothesis 3. These linear estimates give way to the estimates on the nonlinear terms of the function \(G\) in Subsection 5.3. Having then obtained estimates for both the linear and nonlinear terms associated to the function \(G\), we then provide the proof of Theorem 3.2 in Subsection 5.4.
4 Existence and Stability of an Invariant Manifold

Much of this section extends the work of [22, §VII] to our particular situation in infinite dimensions. Throughout this section we assume that Hypothesis 1 and 2 are true, but for convenience we will not explicitly state that they have been assumed in the statements of our results. Subsection 4.1 first deals with the existence of the invariant manifold and then Subsection 4.2 proves the stability.

4.1 Existence of the Invariant Manifold

Let us define the vector space

$$X = \{ \sigma : \ell^1 \to \ell^1 : \sigma(0) = 0, \sup_{\psi \in \ell^1} \|\sigma(\psi)\|_{\infty} < \infty \},$$

along with the associated norm on $X$ given by

$$\|\sigma\|_X := \sup_{\psi \in \ell^1} \|\sigma(\psi)\|_{\infty}. \quad (4.2)$$

Notice that this norm is indeed well-defined because $\ell^1 \subseteq \ell^\infty$ implies that $\|\sigma(\psi)\|_{\infty} < \infty$ for all $\psi \in \ell^1$. Our interest will lie in the following subsets of $X$:

$$\tilde{X}(D, \Delta) = \{ \sigma : \ell^1 \to \ell^1 : \|\sigma\|_X \leq D, \|\sigma(\psi) - \sigma(\tilde{\psi})\|_p \leq \Delta Q_p(\psi - \tilde{\psi}) \forall \psi, \tilde{\psi} \in \ell^1 \}. \quad (4.3)$$

where $D, \Delta > 0$ are constants to be specified shortly. Here the norm on $X$ will simply be used to obtain a contraction of an appropriate mapping whose fixed points are exactly invariant manifolds of (3.11). Hence, what matters is the boundedness and Lipschitz properties of elements of $\tilde{X}(D, \Delta)$ since they will give the desired properties (3.12) of the invariant manifold. We provide the following result.

**Lemma 4.1.** For every $D, \Delta > 0$, $\tilde{X}(D, \Delta)$ is complete with respect to the norm $\| \cdot \|_X$.

**Proof.** Let us fix $D, \Delta > 0$. Then, consider a Cauchy sequence $\{\sigma_n\}_{n=1}^\infty \subset \tilde{X}(D, \Delta)$. The existence of a pointwise limit $\sigma$ converging in the $\ell^\infty$ norm follows in a straightforward way from the fact that $\|\sigma_n(\psi)\|_{\infty} \leq D$ for all $\psi \in \ell^1$. We now wish to show that $\sigma \in \tilde{X}(D, \Delta)$.

For a contradiction, let us assume that there exists $\psi \in \ell^1$ such that $\|\sigma(\psi)\|_{\infty} > D$. Set

$$\varepsilon := \frac{1}{2}(\|\sigma(\psi)\|_{\infty} - D) > 0,$$

and take $N \geq 1$ sufficiently large to guarantee that $\|\sigma_n(\psi) - \sigma(\psi)\|_{\infty} < \varepsilon$. This then gives

$$\|\sigma(\psi)\|_{\infty} \leq \|\sigma_n(\psi) - \sigma(\psi)\|_{\infty} + \|\sigma_n(\psi)\|_{\infty} < \varepsilon + D = \frac{1}{2}(\|\sigma(\psi)\|_{\infty} - D) + D = \frac{1}{2}\|\sigma(\psi)\|_{\infty} + \frac{1}{2}D,$$

where we have used the fact that $\sigma_n \in \tilde{X}(D, \Delta)$ implies that $\|\sigma_n(\psi)\|_{\infty} \leq D$. But then rearranging this expression gives

$$\frac{1}{2}\|\sigma(\psi)\|_{\infty} \leq \frac{1}{2}D \Rightarrow \|\sigma(\psi)\|_{\infty} \leq D,$$

which is a contradiction. This therefore shows that $\|\sigma\|_X \leq D$. 


To show that \(\|\sigma(\psi) - \sigma(\tilde{\psi})\|_p \leq \Delta Q_p(\psi - \tilde{\psi})\) for every \(\psi, \tilde{\psi} \in \ell^1\), we proceed in a nearly identical way to our previous proof showing that \(\|\sigma\|_X \leq D\), and therefore this proof is omitted. We do remark that since \(\sigma_n(0) = 0\) for every \(n \geq 1\), we necessarily have \(\sigma(0) = 0\), and since \(\sigma\) is such that

\[
\|\sigma(\psi) - \sigma(\tilde{\psi})\|_1 \leq \Delta Q_1(\psi - \tilde{\psi})
\]

for every \(\psi, \tilde{\psi} \in \ell^1\), we may take \(\tilde{\psi} = 0\) to find that

\[
\|\sigma(\psi)\|_1 \leq \Delta Q_1(\psi),
\]

for all \(\psi \in \ell^1\). Since Lemma 2.1 details that \(Q_1(\psi) \leq 8\|\psi\|_1 < \infty\), we then have that \(\sigma(\psi) \in \ell^1\) for all \(\psi \in \ell^1\). Therefore, \(\sigma \in \bar{X}(D, \Delta)\), completing the proof.

Now, let us consider arbitrary \(0 < D \leq \frac{a}{4}\) and a function \(\sigma \in \bar{X}(D, \Delta)\). We will denote \(\tilde{\psi}^*(t; \psi^0, \sigma)\) to be the solution of the initial value problem

\[
\begin{cases}
  \dot{\psi} = \alpha G(\sigma(\psi), \psi, \alpha), \\
  \psi(0) = \psi^0,
\end{cases}
\]

(4.4)

where we use the superscript notation \(\psi^0\) so as not to confuse with the subscripts relating to the indices of the lattice \(\mathbb{Z}^2\). Note that we do indeed require the condition \(D \leq \frac{a}{4}\) to guarantee that \(\tilde{\bar{r}}_{i,j}(\alpha) + \sigma_{i,j}(\psi) \geq \frac{a}{4}\), to avoid the singularity in the phase equations when \(\tilde{\bar{r}}_{i,j}(\alpha) + \sigma_{i,j}(\psi) = 0\). We also note that taking \(\psi^0 = 0\) results in the solution \(\psi^*(t; \psi^0, \sigma) = 0\) for all \(t \geq 0\) and \(\sigma \in \bar{X}(D, \Delta)\) since \(\sigma(0) = 0\) and \(G(0, 0, \alpha) = 0\).

**Lemma 4.2.** There exists a constant \(C_1 > 0\) such that for every \(D \in (0, \frac{a}{4}]\), \(\Delta \in (0, 1]\), \(\sigma, \tilde{\sigma} \in \bar{X}(D, \Delta)\), \(\psi, \tilde{\psi} \in \ell^1\), and \(p \in [1, \infty]\) we have the following:

\[
Q_p(\psi^*(t; \psi, \sigma) - \psi^*(t; \tilde{\psi}, \sigma)) \leq e^{\alpha C_1 t} Q_p(\psi - \tilde{\psi}),
\]

(4.5a)

\[
\|\psi^*(t; \psi, \sigma) - \psi^*(t; \psi, \tilde{\sigma})\|_\infty \leq e^{\alpha C_1 t} \|\sigma - \tilde{\sigma}\|_X,
\]

(4.5b)

for all \(t \in \mathbb{R}\).

**Proof.** We will only prove the inequalities for \(t \geq 0\), since the proof for \(t < 0\) is handled in a nearly identical way by introducing the temporal transformation \(t \to -t\) and taking advantage of the fact that the differential equation is autonomous. Then to begin, recall that \(G(0, 0, \alpha) = 0\), by definition of \(s\) and \(\psi\). Furthermore, for each \((i, j) \in \mathbb{Z}^2\) we have that \(G_{i,j}(s, \psi, \alpha)\) vanishes when \(s_{i,j} = 0\), \(s_{i,j} = 0\) and \((\psi_{i,j} - \tilde{\psi}_{i,j}) = 0\), for all \((i', j')\). Hence, from the fact that \(G_{i,j}(s, \psi, \alpha)\) is globally Lipschitz in \(\psi\), uniformly in \((i, j)\), and since \(\|\sigma(\psi)\|_\infty\) is uniformly bounded by \(D \leq \frac{a}{4}\), we find that there exists a \(C > 0\), uniform in \(D \in (0, \frac{a}{4}]\), such that

\[
|G_{i,j}(\sigma(\psi), \psi, \alpha) - G_{i,j}(\tilde{\sigma}(\tilde{\psi}), \tilde{\psi}, \alpha)| \leq C \left( |\sigma_{i,j}(\psi) - \tilde{\sigma}_{i,j}(\tilde{\psi})| + \sum_{i', j'} |\sigma_{i', j'}(\psi) - \tilde{\sigma}_{i', j'}(\tilde{\psi})| \right)
\]

\[
+ \sum_{i', j'} |(\psi_{i', j'} - \tilde{\psi}_{i,j}) - (\tilde{\psi}_{i', j'} - \tilde{\tilde{\psi}}_{i,j})|,
\]

(4.6)
for all $\alpha \in [0, \alpha^*]$.

Now starting with the first bound we wish to prove, begin by taking $\sigma = \tilde{\sigma}$. Then, from Lemma 2.2 we have that

$$\left( \sum_{(i,j)\in Z^2} \left( \sum_{i',j'} |(\psi_{i',j'} - \psi_{i,j}) - (\tilde{\psi}_{i',j'} - \tilde{\psi}_{i,j})| \right) \right)^{\frac{1}{p}} \leq 4Q_p(\psi - \tilde{\psi}),$$

for all $p \in [1, \infty)$. Hence, we have that

$$\|G(\sigma(\psi), \psi, \alpha) - G(\sigma(\tilde{\psi}), \tilde{\psi}, \alpha)\|_p \leq 5C\|\sigma(\psi) - \sigma(\tilde{\psi})\|_p + 4CQ_p(\psi - \tilde{\psi}),$$

$$\leq 5C\Delta Q_p(\psi - \tilde{\psi}) + 4CQ_p(\psi - \tilde{\psi}),$$

$$\leq (5\Delta + 1)CQ_p(\psi - \tilde{\psi}),$$

$$\leq 6CQ_p(\psi - \tilde{\psi}),$$

$$\leq 9CQ_p(\psi - \tilde{\psi}),$$

so that

$$\|G(\sigma(\psi), \psi, \alpha) - G(\sigma(\tilde{\psi}), \tilde{\psi}, \alpha)\|_p \leq 9CQ_p(\psi - \tilde{\psi}), \quad (4.7)$$

for all $p \in [1, \infty]$ and $\psi, \tilde{\psi} \in \ell^1$.

Then, if $\psi^*(t; \psi, \sigma)$ is a solution of (4.4), it therefore satisfies the integral form equation

$$\psi^*(t; \psi, \sigma) = \psi + \int_0^t G(\sigma(\psi^*(u; \psi, \sigma)), \psi^*(u; \psi, \sigma), \alpha)du. \quad (4.8)$$

Hence, using this integral formulation we obtain

$$Q_p(\psi^*(t; \psi, \sigma) - \psi^*(t; \tilde{\psi}, \sigma)) \leq Q_p(\psi - \tilde{\psi})$$

$$+ \int_0^t \alpha Q_p(G(\sigma(\psi^*(u; \psi, \sigma)), \psi^*(u; \psi, \sigma), \alpha) - G(\psi^*(u; \psi, \sigma), \psi^*(u; \tilde{\psi}, \sigma), \alpha))du$$

$$\leq Q_p(\psi - \tilde{\psi}) + \int_0^t 8\alpha \|\psi^*(u; \psi, \sigma) - \psi^*(u; \tilde{\psi}, \sigma)\|_p du,$$

$$\leq Q_p(\psi - \tilde{\psi}) + \int_0^t 72C\alpha Q_p(\psi^*(u; \psi, \sigma) - \psi^*(u; \tilde{\psi}, \sigma))du,$$

where we have applied the righthand bound of Lemma 2.1 followed by the bound (4.7) in the final two steps, respectively. Then, using Gronwall’s inequality we obtain

$$Q_p(\psi^*(t; \psi, \sigma) - \psi^*(t; \tilde{\psi}, \sigma)) \leq e^{72C\alpha t}Q_p(\psi - \tilde{\psi}).$$
This proves the first bound stated in the lemma.

Now, using the bound \([4.6]\) again we have
\[
|G_{i,j}(\sigma(\psi), \psi, \alpha) - G_{i,j}(\tilde{\sigma}(\psi), \tilde{\psi}, \alpha)| \leq 5C\|\sigma(\psi) - \sigma(\tilde{\psi})\|_{\infty} + C\sum_{i,j',j''} |\psi_{i,j'} - \tilde{\psi}_{i,j''}|,
\]
\[
\leq 5C\|\sigma - \tilde{\sigma}\|_{X} + 4C\|\psi - \tilde{\psi}\|_{\infty}
\]
for all \((i, j) \in \mathbb{Z}^{2}\). Hence, using the integral formulation \([4.8]\), we arrive at
\[
|\psi_{i,j}^*(t; \psi, \sigma) - \psi_{i,j}^*(t; \tilde{\psi}, \tilde{\sigma})| \leq \alpha \int_{0}^{t} |G(\sigma(\psi^*(u; \psi, \tilde{\sigma})), \psi^*(u; \psi, \sigma), \alpha) - G(\psi^*(u; \psi, \tilde{\sigma}), \psi^*(u; \tilde{\psi}, \tilde{\sigma}), \alpha)|du
\]
\[
\leq 5C\alpha \int_{0}^{t} \|\sigma - \tilde{\sigma}\|_{X} + \|\psi^*(u; \psi, \sigma) - \psi^*(u; \tilde{\psi}, \tilde{\sigma})\|_{\infty} du
\]
\[
\leq 5C\alpha\|\sigma - \tilde{\sigma}\|_{X}t + 4C\alpha \int_{0}^{t} \|\psi^*(u; \psi, \sigma) - \psi^*(u; \tilde{\psi}, \tilde{\sigma})\|_{\infty} du.
\]
Taking the supremum over all \((i, j) \in \mathbb{Z}^{2}\) we arrive at
\[
\|\psi^*(t; \psi, \sigma) - \psi^*(t; \psi, \tilde{\sigma})\|_{\infty} \leq 5C\alpha\|\sigma - \tilde{\sigma}\|_{X}t + 4C\alpha \int_{0}^{t} \|\psi^*(u; \psi, \sigma) - \psi^*(u; \tilde{\psi}, \tilde{\sigma})\|_{\infty} du.
\]
We now apply Gronwall’s inequality to arrive at the bound
\[
\|\psi^*(t; \psi, \sigma) - \psi^*(t; \psi, \tilde{\sigma})\|_{\infty} \leq 5C\alpha e^{4C\alpha t}\|\sigma - \tilde{\sigma}\|_{X}.
\]
Finally, there exists \(C' > 0\) such that
\[
5C\alpha e^{4C\alpha t} \leq e^{a\alpha C't},
\]
for all \(t \geq 0\) and \(\alpha \in [0, \alpha^*]\). This completes the proof. \(\square\)

Our goal is now to obtain a mapping acting on the space \(\tilde{X}(D, \Delta)\), for \(D, \Delta > 0\) appropriately chosen, so that a fixed point is exactly an invariant manifold for the differential equations \([3.11]\). To begin, let us rewrite
\[
\dot{s} = F(s, \psi, \alpha),
\]
as
\[
\dot{s} = a\lambda'(a)s + [F(s, \psi, \alpha) - a\lambda'(a)s].
\]
One should recall that the derivative of \(F\) with respect to \(s\) is \(D_{s}\sigma(0, 0, \alpha) = a\lambda'(a)I\), where \(I\) is the identity mapping. We recall that \(a\lambda'(a) < 0\) from Hypothesis \([1]\). Then, using the variation of constants formula we arrive at
\[
s(t) = e^{a\lambda'(a)(t-t_0)}s(t_0) + \int_{t_0}^{t} e^{a\lambda'(a)(t-u)}[F(s(u), \psi(u), \alpha) - a\lambda'(a)s(u)]du.
\]
Assuming that \(s\) belongs to \(\tilde{X}(D, \Delta)\) for all \(t\), it is bounded and we can therefore take \(t_0 \to -\infty\) to arrive at
\[
s(t) = \int_{-\infty}^{t} e^{a\lambda'(a)(t-u)}[F(s(u), \psi(u), \alpha) - a\lambda'(a)s(u)]du.
\]
15
Finally, taking \( t = 0 \) allows one to define a mapping \( T \) with domain \( \tilde{X}(D, \Delta) \), for appropriately chosen \( D, \Delta > 0 \), given by

\[
T_1 \sigma(\psi) = \int_{-\infty}^{0} e^{-a\lambda^1(a)u} \left[ F(\psi^*(u; \psi, \sigma), \psi^*(u; \psi, \sigma), \alpha) - a\lambda'(a)\sigma(\psi^*(u; \psi, \sigma)) \right] du. \tag{4.9}
\]

Fixed points of \( T_1 \) are exactly invariant manifolds of the full system (3.11). To understand this mapping, \( \psi \in \ell^1 \) is an initial condition of the flow governed by (4.4) using the function \( \sigma \). This solution, denoted \( \psi^*(t; \psi, \sigma) \), is then put into the radial component equation and we describe the flow of the radial component. If this flow matches the flow governed by the flow of the original input function \( \sigma \), we have indeed obtained a flow-invariant invariant manifold for (4.4). For the ease of notation we will define

\[
\tilde{F}(s, \psi, \alpha) = F(s, \psi, \alpha) - a\lambda'(a)s, \tag{4.10}
\]

so that we can write

\[
T_1 \sigma(\psi) = \int_{-\infty}^{0} e^{-a\lambda^1(a)u} \tilde{F}(\psi^*(u; \psi, \sigma), \psi^*(u; \psi, \sigma), \alpha) du.
\]

Notice that \( \tilde{F}(0, \psi, \alpha) = F(0, \psi, \alpha) \) and therefore \( \tilde{F}(0, 0, \alpha) = 0 \) for all \( \alpha \in [0, \alpha^*] \).

**Lemma 4.3.** There exists \( \alpha_{X,1} > 0 \) such that for each \( \alpha \in [0, \alpha_{X,1}] \), \( T_1 : \tilde{X}(\sqrt{\alpha}, \sqrt{\alpha}) \to \tilde{X}(\sqrt{\alpha}, \sqrt{\alpha}) \) is well-defined.

**Proof.** We break this proof into three major components to show that for appropriately chosen \( D, \Delta > 0 \) and sufficiently small \( \alpha > 0 \) we can guarantee that \( T_1 \sigma(0) = 0 \), \( \|T_1 \sigma\|_X \leq D \) and

\[
\|T_1 \sigma(\psi) - T_1 \sigma(\tilde{\psi})\|_P \leq \Delta Q_p(\psi - \tilde{\psi})
\]

for all \( \psi, \tilde{\psi} \in \ell^1, \sigma \in \tilde{X}(\sqrt{\alpha}, \sqrt{\alpha}) \), and \( p \in [1, \infty] \). Throughout this proof we will always assume \( \alpha > 0 \) is taken to satisfy: \( \alpha \leq \min\{\alpha^*, (\frac{\alpha}{2})^2, 1\} \), so that our choices \( D = \sqrt{\alpha} \) and \( \Delta = \sqrt{\alpha} \) satisfy the assumptions of Lemma 4.2.

\( T_1 \sigma(0) = 0 \): Recall that taking \( \psi = 0 \) results in \( \psi^*(t; 0, \sigma) = 0 \) for all \( \sigma \) since \( \sigma(0) = 0 \). Then, evaluating \( T_1 \sigma(0) \) gives

\[
T_1 \sigma(0) = \int_{-\infty}^{0} e^{-a\lambda^1(a)u} \tilde{F}(0, 0, \alpha) du = 0,
\]

since \( \tilde{F}(0, 0, \alpha) = 0 \) for all \( \alpha \in [0, \alpha^*] \).

\( \|T_1 \sigma\|_X \leq D \): To begin, we consider arbitrary \( \psi \in \ell^1 \) and recall that

\[
\tilde{\sigma}_{i,j}(\sigma(\psi^*(t; \psi, \sigma)), \psi^*(t; \psi, \sigma), \alpha)
= \alpha \sum_{i', j'} \left[ (\tilde{r}_{i', j'}(\alpha) + \tilde{\sigma}_{i', j'}(\psi^*(t; \psi, \sigma))) \cos(\tilde{\theta}_{i', j'}(\alpha) + \tilde{\psi}_{i', j'}(t; \psi, \sigma) - \tilde{\theta}_{i,j}(\alpha) - \tilde{\psi}_{i,j}(t; \psi, \sigma))
- (\tilde{r}_{i,j}(\alpha) + \tilde{\sigma}_{i,j}(\psi^*(t; \psi, \sigma))) \right] + (\tilde{r}_{i,j}(\alpha) + \tilde{\sigma}_{i,j}(\psi^*(t; \psi, \sigma))) \lambda(\tilde{r}_{i,j}(\alpha) + \tilde{\sigma}_{i,j}(\psi^*(t; \psi, \sigma)) - a\lambda'(a)\sigma_{i,j}(\psi^*(t; \psi, \sigma))). \tag{4.11}
\]
For convenience, we will break down the bounds of this term into separate parts. First, we have

\[ |(\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma))) \cos(\tilde{\theta}_{i,j}(\alpha) + \psi_{i,j}(t; \psi, \sigma) - \tilde{\psi}_{i,j}(\alpha) - \tilde{\psi}_{i,j}(t; \psi, \sigma))| \leq \frac{3a}{2} + \sqrt{\alpha}, \]

since our definition of \( \alpha^* > 0 \) implies that \( |\tilde{r}_{i,j}(\alpha)| \leq \frac{3a}{2} \) and \( |\sigma_{i,j}(\psi^*(u; \psi, \sigma)| \leq \sqrt{\alpha} \) since \( \sigma \in \bar{X}(\sqrt{\alpha}, \sqrt{\alpha}) \). Similarly,

\[ |\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma))| \leq \frac{3a}{2} + \sqrt{\alpha}. \]

Then, we use the fact that \( \lambda(a) = 0 \) to apply Taylor’s Theorem to see that there exists a constant \( C_\lambda > 0 \), independent of \( D = \sqrt{\alpha} \leq \frac{q}{4} \), so that

\[ |(\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma)))\lambda(\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma))) - a\lambda'(a)\sigma_{i,j}(\psi^*(t; \psi, \sigma))| \leq C_\lambda |\lambda(\sigma_{i,j}(\psi^*(t; \psi, \sigma)))| \]

\[ \leq C_\lambda a\lambda(\sigma_{i,j}(\psi^*(t; \psi, \sigma))) + |\tilde{r}_{i,j}(\alpha) - a| \]

\[ = (1 + C_r)C_\lambda \alpha \]

(4.12)

where \( C_r > 0 \) is the constant guaranteed by (3.8). Then, using these inequalities we therefore return to (4.11) to see that

\[ |\tilde{F}_{i,j}(\sigma(t; \psi, \sigma), \psi^*(t; \psi, \sigma), \alpha)| \leq 4\alpha \left( \frac{3a}{2} + \sqrt{\alpha} \right) + 4\alpha \left( \frac{3a}{2} + \sqrt{\alpha} \right) + (1 + C_r)C_\lambda \alpha \]

\[ = 8\alpha \left( \frac{3a}{2} + \sqrt{\alpha} \right) + (1 + C_r)C_\lambda \alpha \]

\[ \leq 8\alpha \left( \frac{3a}{2} + \frac{a}{4} \right) + (1 + C_r)C_\lambda \alpha \]

\[ \leq (14a + C_r + C_\lambda) \alpha, \]

for all \((i, j) \in \mathbb{Z}^2\), since we have assumed \( \sqrt{\alpha} \leq \frac{q}{4} \). Therefore, recalling that \( a\lambda'(a) < 0 \), this then implies that for any \( \psi \in \ell^1 \) we have

\[ \|T_1\sigma(\psi)\|_\infty \leq (14a + C_\lambda + C_rC_\lambda) \alpha \int_{-\infty}^{0} e^{-a\lambda'(a)u} du \]

\[ \leq \frac{-1}{a\lambda'(a)} \left[ 14a + C_\lambda + C_rC_\lambda \right] \alpha. \]

Taking

\[ \alpha \leq \min \left\{ \alpha^*, \left( \frac{a}{4} \right)^2, 1, \left( \frac{a\lambda'(a)}{14a + C_\lambda + C_rC_\lambda} \right)^2 \right\}, \]

gives that \( \|T_1\sigma(\psi)\|_\infty \leq \sqrt{\alpha} \), for all \( \psi \in \ell^1 \). Then taking the supremum over all \( \psi \in \ell^1 \) we have \( \|T_1\sigma\|_\infty \leq \sqrt{\alpha} \), as required.

\[ \|T_1\sigma(\psi) - T_1\sigma(\tilde{\psi})\|_p \leq \Delta Q_p(\psi - \tilde{\psi}) \] is the graph of any \( \psi \)

Begin by fixing \( \sigma \in \bar{X}(\sqrt{\alpha}, \sqrt{\alpha}), \psi, \tilde{\psi} \in \ell^1 \). We again use the form (4.11) and break the bounds into smaller pieces as in the proof of the previous bound.
To begin, we use the uniform boundedness of cosine and its derivatives to obtain
\[
|\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma))| \cos(\tilde{\theta}_{i,j}(\alpha) + \psi_{i,j}(t; \psi, \sigma) - \tilde{\theta}_{i,j}(\alpha) - \psi_{i,j}(t; \psi, \sigma))
\]
\[-(\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \tilde{\psi}, \sigma))) \cos(\tilde{\theta}_{i,j}(\alpha) + \psi_{i,j}(t; \tilde{\psi}, \sigma) - \tilde{\theta}_{i,j}(\alpha) - \psi_{i,j}(t; \tilde{\psi}, \sigma))| \leq |\sigma_{i,j}(\psi^*(t; \psi, \sigma)) - \sigma_{i,j}(\psi^*(t; \tilde{\psi}, \sigma))| + |(\psi_{i,j}(t; \psi, \sigma) - \psi_{i,j}(t; \psi, \sigma)) - (\psi_{i,j}(t; \tilde{\psi}, \sigma) - \psi_{i,j}(t; \tilde{\psi}, \sigma))|.
\]
Then, trivially we have
\[
|((\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma))) - (\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \tilde{\psi}, \sigma)))| = |\sigma_{i,j}(\psi^*(t; \psi, \sigma)) - \sigma_{i,j}(\psi^*(t; \tilde{\psi}, \sigma))|,
\]
which we point out for the sake of completeness. And finally, for all $\sigma \in \tilde{X}(\sqrt{\alpha}, \sqrt{\alpha})$, there exists a $C'_\alpha > 0$, independent of $0 \leq \alpha \leq \frac{4}{3}$, so that
\[
|\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma)) - \lambda(\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma))) - a\lambda'(\alpha)| \leq C'_\alpha |\sigma_{i,j}(\psi^*(t; \psi, \sigma)) - \sigma_{i,j}(\psi^*(t; \tilde{\psi}, \sigma))|,
\]
since the function $\lambda(\tilde{r}_{i,j}(\alpha) + x) + (\tilde{r}_{i,j}(\alpha) + x) \lambda(\tilde{r}_{i,j}(\alpha) + x) - a\lambda(\alpha)|\sigma_{i,j}(\psi^*(t; \psi, \sigma)) - \sigma_{i,j}(\psi^*(t; \tilde{\psi}, \sigma))| = 0$ for all $(i, j) \in \mathbb{Z}^2$.

Now, we use these three previous bounds and the form (4.11) to see that
\[
|\tilde{F}_{i,j}(\sigma^*(t; \psi, \sigma), \psi^*(t; \psi, \sigma), \alpha) - \tilde{F}_{i,j}(\sigma^*(t; \tilde{\psi}, \sigma), \psi^*(t; \tilde{\psi}, \sigma), \alpha)|
\leq \alpha \sum_{i'j'} |\sigma_{i',j'}(\psi^*(t; \psi, \sigma)) - \sigma_{i',j'}(\psi^*(t; \tilde{\psi}, \sigma))| + 4\alpha |\sigma_{i,j}(\psi^*(t; \psi, \sigma)) - \sigma_{i,j}(\psi^*(t; \tilde{\psi}, \sigma))|
\]
\[
+ \alpha \sum_{i'j'} |(\psi_{i',j'}(t; \psi, \sigma) - \psi_{i',j'}(t; \tilde{\psi}, \sigma)) - (\psi_{i',j'}(t; \psi, \sigma) - \psi_{i',j'}(t; \tilde{\psi}, \sigma))|
\]
\[
+ C'_\alpha |\sigma_{i,j}(\psi^*(t; \psi, \sigma)) - \sigma_{i,j}(\psi^*(t; \tilde{\psi}, \sigma))|,
\]
for all $(i, j) \in \mathbb{Z}^2$. Then, for all $p \in [1, \infty)$, using Lemma 2.2 we obtain
\[
\|\tilde{F}(\sigma^*(t; \psi, \sigma), \psi^*(t; \psi, \sigma), \alpha) - \tilde{F}(\sigma^*(t; \tilde{\psi}, \sigma), \psi^*(t; \tilde{\psi}, \sigma), \alpha)\|_p
\leq 8\alpha \|\sigma^*(t; \psi, \sigma) - \sigma^*(t; \tilde{\psi}, \sigma)\|_p + 4\alpha(\sigma^*(t; \psi, \sigma) - \sigma^*(t; \tilde{\psi}, \sigma))
\]
\[
+ C'_\alpha \|\sigma^*(t; \psi, \sigma) - \sigma^*(t; \tilde{\psi}, \sigma)\|_p
\leq (8\alpha^\frac{2}{3} + 4\alpha + C'_\alpha) Q_p(\psi^*(t; \psi, \sigma) - \psi^*(t; \tilde{\psi}, \sigma))
\leq (13 + C'_\alpha)\alpha e^{\alpha C'_1|t|} Q_p(\psi - \tilde{\psi}),
\]
where we have used the facts that $\sigma \in \tilde{X}(\sqrt{\alpha}, \sqrt{\alpha})$ and $\sqrt{\alpha} \leq 1$, as well as applied (4.5a) from Lemma 4.2 with the constant $C_1 > 0$. Taking $\alpha \leq \frac{-a\lambda'(a)}{2e^{C_1}}$ guarantees that
\[
\alpha C_1 + a\lambda'(a) \leq \frac{a\lambda'(a)}{2} < 0,
\]
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and hence,

\[ ||T_1\sigma(\psi) - T_1\sigma(\tilde{\psi})||_p \leq (9 + C'_\lambda)\alpha Q_p(\psi - \tilde{\psi}) \int_{-\infty}^0 e^{-(\alpha C_1 + a\lambda'(a))u} du \]

\[ = \left(\frac{9 + C'_\lambda}{-(\alpha C_1 + a\lambda'(a))}\right)\alpha Q_p(\psi - \tilde{\psi}) \]

\[ \leq \left(\frac{18 + 2C'_\lambda}{\alpha\lambda'(a)}\right)\alpha Q_p(\psi - \tilde{\psi}). \]

Therefore, taking

\[ \alpha \leq \min \left\{ \alpha^*, \left(\frac{a}{4}\right)^2, 1, \frac{-a\lambda'(a)}{2C'_1}, \left(\frac{a\lambda'(a)}{18 + 2C'_\lambda}\right)^2 \right\}, \]

provides that \( ||T_1\sigma(\psi) - T_1\sigma(\tilde{\psi})||_p \leq \sqrt{\alpha} Q_p(\psi - \tilde{\psi}) \) for all \( \psi, \tilde{\psi} \in \ell^1 \) and \( p \in [1, \infty) \). The case when \( p = \infty \) follows in exactly the same way, and is committed.

In closing, we can define \( \alpha_{X,1} \) as

\[ \alpha_{X,1} := \min \left\{ \alpha^*, \left(\frac{a}{4}\right)^2, 1, \frac{-a\lambda'(a)}{14a + C_\lambda C'_\lambda} + \frac{a\lambda'(a)}{2C'_1}, \left(\frac{a\lambda'(a)}{18 + 2C'_\lambda}\right)^2 \right\} \]

so that for all \( \alpha \in [0, \alpha_{X,1}] \) we have that \( T_1 : \dot{X}(\sqrt{\alpha}, \sqrt{\alpha}) \to \dot{X}(\sqrt{\alpha}, \sqrt{\alpha}) \). This concludes the proof. \( \square \)

**Lemma 4.4.** There exists \( \alpha_{X,2} > 0 \) such that for each \( \alpha \in [0, \alpha_{X,2}] \), \( T_1 : \dot{X}(\sqrt{\alpha}, \sqrt{\alpha}) \to \dot{X}(\sqrt{\alpha}, \sqrt{\alpha}) \) is a contraction with contraction constant at most \( \frac{1}{2} \).

**Proof.** This proof proceeds by applying very similar manipulations to that of the previous lemma to show that \( T_1 \) is well-defined, and therefore we will omit some details which are redundant. Furthermore, we will always consider \( \alpha \leq \alpha_{X,1} \), so that the conclusion of Lemma 4.3 holds and that our choices \( D = \sqrt{\alpha} \) and \( \Delta = \sqrt{\alpha} \) satisfy the assumptions of Lemma 4.2.

To begin, let us consider \( \sigma, \tilde{\sigma} \in \dot{X}(\sqrt{\alpha}, \sqrt{\alpha}) \). Then, following the manipulations in (4.13) we obtain the similar bound

\[
|\tilde{F}_{i,j}(\sigma^*(t; \psi, \sigma), \psi^*(t; \psi, \sigma), \alpha) - \tilde{F}_{i,j}(\tilde{\sigma}(\psi^*(t; \psi, \tilde{\sigma}), \psi^*(t; \psi, \tilde{\sigma}), \alpha)|
\leq \alpha \sum_{i',j'} |\sigma_{i',j'}(\psi^*(t; \psi, \sigma)) - \tilde{\sigma}_{i',j'}(\psi^*(t; \psi, \tilde{\sigma}))| + 4\alpha |\sigma_{i,j}(\psi^*(t; \psi, \sigma)) - \tilde{\sigma}_{i,j}(\psi^*(t; \psi, \tilde{\sigma}))|
\]

\[ + \alpha \sum_{i',j'} |(\psi_{i',j'}^*(t; \psi, \sigma) - \psi_{i,j}^*(t; \psi, \sigma)) + (\psi_{i',j'}^*(t; \psi, \tilde{\sigma}) - \psi_{i,j}^*(t; \psi, \tilde{\sigma}))| + C'_\lambda \alpha |\sigma_{i,j}(\psi^*(t; \psi, \sigma)) - \tilde{\sigma}_{i,j}(\psi^*(t; \psi, \tilde{\sigma}))|, \]

(4.14)

for all \( (i, j) \in \mathbb{Z}^2 \), and we recall that \( C'_\lambda > 0 \) is the constant used in the proof of Lemma 4.3 for which

\[
|\lambda(\tilde{r}_{i,j}(\alpha) + x) + (\tilde{r}_{i,j}(\alpha) + x)\lambda'(\tilde{r}_{i,j}(\alpha) + x) - a\lambda'(a)| \leq C'_\lambda \alpha
\]
for all $|x| \leq \sqrt{\alpha}$. Then, using \[4.14\] we can take the supremum over all $(i, j) \in \mathbb{Z}^2$ to get
\[
\|\tilde{F}(\psi^*(t; \psi, \sigma), \psi^*(t; \psi, \sigma), \alpha) - \tilde{F}(\tilde{\sigma}(t; \psi, \tilde{\sigma}), \psi^*(t; \psi, \tilde{\sigma}), \alpha)\|_\infty \\
\leq (8 + C'_\lambda)\alpha\|\sigma - \tilde{\sigma}\|_\infty + 4\alpha\|\psi^*(t; \psi, \sigma) - \psi^*(t; \psi, \tilde{\sigma})\|_\infty \\
\leq (8 + C'_\lambda)\alpha\|\sigma - \tilde{\sigma}\|_\infty + 4\alpha e^{\lambda C_1|t|}\|\sigma - \tilde{\sigma}\|_X \\
\leq (12 + C'_\lambda)\alpha e^{\lambda C_1|t|}\|\sigma - \tilde{\sigma}\|_X.
\]

Therefore, taking $\alpha \leq \frac{-a\lambda'(a)}{2C_1}$ guarantees that
\[
\alpha C_1 + a\lambda'(a) \leq \frac{a\lambda'(a)}{2} < 0,
\]
and hence,
\[
\|T_1\sigma(\psi) - T\tilde{\sigma}(\psi)\|_\infty \leq (12 + C'_\lambda)\alpha\|\sigma - \tilde{\sigma}\|_X \int_{-\infty}^{0} e^{-(\alpha C_1 + a\lambda'(a))u} du \\
\leq -\left(\frac{12 + C'_\lambda}{\alpha C_1 + a\lambda'(a)}\right)\alpha\|\sigma - \tilde{\sigma}\|_X \\
\leq -\left(\frac{24 + 2C'_\lambda}{a\lambda'(a)}\right)\alpha\|\sigma - \tilde{\sigma}\|_X.
\]

Then, taking
\[
\alpha_{X,2} := \min\left\{\alpha^*, \alpha_{X,1}, \frac{-a\lambda'(a)}{2C_1}, \frac{-a\lambda'(a)}{48 + 4C'_\lambda}\right\}
\]
gives $\|T_1\sigma(\psi) - T_1\tilde{\sigma}(\psi)\|_\infty \leq \frac{1}{2}\|\sigma - \tilde{\sigma}\|_X$ for all $\psi \in \ell^1$. Taking the supremum over $\psi \in \ell^1$ shows that $T_1$ is a contraction with contraction constant at most $\frac{1}{2}$, concluding the proof. \hfill \Box

Lemma \[4.4\] gives that for each $\alpha \in [0, \alpha_{X,2}]$ there exists a unique fixed point of the mapping $T_1$, simply denoted $\sigma(\psi, \alpha) \in \tilde{X}(\sqrt{\alpha}, \sqrt{\alpha})$. Recall that the properties of $\tilde{X}(\sqrt{\alpha}, \sqrt{\alpha})$ therefore imply that
\[
\sigma(0, \alpha) = 0, \\
\|\sigma(\psi, \alpha)\|_\infty \leq \sqrt{\alpha}, \\
\|\sigma(\psi, \alpha) - \sigma(\tilde{\psi}, \alpha)\|_p \leq \sqrt{\alpha}Q_p(\psi - \tilde{\psi}),
\]
for all $\psi, \tilde{\psi} \in \ell^1$ and $p \in [1, \infty]$. As previously stated, this fixed point corresponds to an invariant manifold of the differential equation \[3.11\], therefore attaining our objective for this subsection.

### 4.2 Stability of the Invariant Manifold

This subsection proceeds in a similar way to the previous subsection in that we apply a bootstrapping argument to an appropriate mapping to determine the stability of the invariant manifold. In fact, much of this section follows the proof of the Stable Manifold Theorem, but since we are working in infinite dimensions some extra attention must be paid to certain aspects of the problem. Recall that in the
previous section we determined the existence of an invariant manifold for the system (3.11), which we write as a function of the phase variable: \( \sigma(\psi, \alpha) \), for \( \alpha \geq 0 \) sufficiently small.

Now, to understand the decay of perturbations from the invariant manifold, we write \( s = \sigma(\psi, \alpha) + \rho \), where \( \rho \) captures the deviation of \( s \) from the invariant manifold. For a fixed \( \delta > 0 \), let us consider the spaces

\[
Y(\delta) = \{ \rho : [0, \infty) \to \ell^1 : \sup_{t \geq 0} \| \rho(t) \|_1 \leq 2\delta e^{\frac{a\lambda'(\alpha)}{2} t} \}
\]

along with associated norm

\[
\| \rho(t) \|_Y := \sup_{t \in [0, \infty)} \| \rho(t) \|_1.
\]

Here we recall that \( a\lambda'(\alpha) < 0 \), and hence the \( \ell^1 \)-norm of elements in \( Y \) decay exponentially in \( t \). We present the following lemma.

**Lemma 4.5.** For every \( \delta > 0 \), \( Y(\delta) \) is a complete with respect to the norm \( \| \cdot \|_Y \).

**Proof.** Let us begin by fixing \( \delta > 0 \). Let us take \( \{ \rho_n \}_{n=1}^\infty \subset Y(\delta) \) to be a Cauchy sequence. Then, by definition we have

\[
\| \rho_n(t) \|_1 \leq \| \rho_n \|_Y \leq 2\delta e^{\frac{a\lambda'(\alpha)}{2} t} \leq 2\delta,
\]

for all \( t \in [0, \infty) \) and \( n \geq 1 \). Then, as in the proof of Lemma 4.1, uniformity of the norm in \( t \geq 0 \) implies the existence of a pointwise limit, denoted \( \rho(t) \), so that \( \rho_n(t) \to \rho(t) \) in \( \ell^1 \) for all \( t \in [0, \infty) \). Furthermore, the proof of the decay of the \( \ell^1 \)-norm of \( \rho(t) \) in \( t \) to ensure \( \rho \in Y(\delta) \) follows through nearly identical arguments to those laid out in Lemma 4.1. 

Throughout this subsection, as in the former, we will use the constant \( \alpha_{X,2} > 0 \) to represent the maximal value of \( \alpha \) for which the invariant manifold \( \sigma(\cdot, \alpha) \) exists. Following as in the previous subsection, we will let \( \psi^*(t; \psi^0, \sigma + \rho) \) denote that solution to the initial value problem

\[
\begin{aligned}
\dot{\psi} &= \alpha G(\sigma(\psi, \alpha) + \rho(t), \psi, \alpha), \\
\psi(0) &= \psi^0.
\end{aligned}
\]

(4.16)

Note again that we require the condition \( \rho \in \bar{Y}(\delta) \) for \( \delta \leq \frac{\delta}{16} \) to guarantee that \( \bar{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi, \alpha) + \rho(t) \geq \frac{\alpha}{16} \), to avoid the singularity in the phase equations when \( \bar{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi, \alpha) + \rho(t) = 0 \). This leads to our first result which is similar to Lemma 4.2.

**Lemma 4.6.** There exists a constant \( C_2 > 0 \) such that for all \( \delta \in [0, \frac{\alpha}{16}] \), \( \rho, \tilde{\rho} \in \bar{Y}(\delta) \), \( \psi \in \ell^1 \) and \( \alpha \in [0, \alpha_{X,2}] \) sufficiently small we have

\[
\| \psi^*(t; \psi, \sigma + \rho) - \psi^*(t; \psi, \sigma + \tilde{\rho}) \|_1 \leq e^{\alpha C_2 t} \| \rho - \tilde{\rho} \|_Y,
\]

(4.17)

for all \( t \geq 0 \).

**Proof.** This proof proceeds in a similar way to that of (4.5b) in Lemma 4.2 and therefore many details will be omitted. Nearly identical manipulations to those undertaken in the proof of Lemma 4.2 yield
for all \( \rho, \tilde{\rho} \), of from both the properties of the invariant manifold and the inequalities of Lemma 2.1. Therefore, we omit these final steps due to their redundancy and complete the proof of (4.5b).

Now, taking \( \sigma, \alpha \) such that

\[
|G_{i,j}(\sigma(\psi, \alpha) + \rho(t), \psi, \alpha) - G_{i,j}(\tilde{\sigma}(\psi, \alpha) + \tilde{\rho}(t), \tilde{\psi}, \alpha)| \leq C \left( |\rho_{i,j}(t) + \tilde{\rho}_{i,j}(t)| + \sum_{i',j'} |\rho_{i',j'}(t) + \tilde{\rho}_{i',j'}(t)| \right)
\]

\[
+ |\sigma_{i,j}(\psi, \alpha) + \sigma_{i,j}(\tilde{\psi}, \alpha)| + \sum_{i',j'} |\sigma_{i',j'}(\psi, \alpha) + \sigma_{i',j'}(\tilde{\psi}, \alpha)|
\]

\[
+ |\psi_{i,j} - \tilde{\psi}_{i,j}| + \sum_{i',j'} |\psi_{i',j'} - \tilde{\psi}_{i',j'}|,
\]

for all \( \rho, \tilde{\rho} \in \tilde{Y}(\delta) \) with \( \delta \in [0, \frac{\alpha}{10}], \psi, \tilde{\psi} \in \ell^1 \) and \( \alpha \) taken sufficiently small. Then, taking the sum over all \( (i, j) \in \mathbb{Z}^2 \) we obtain

\[
\|G(\sigma(\psi, \alpha) + \rho(t), \psi, \alpha) - G(\tilde{\sigma}(\psi, \alpha) + \tilde{\rho}(t), \tilde{\psi}, \alpha)\|_1 \leq 5C\|\rho(t) - \tilde{\rho}(t)\|_1 + 5C\|\sigma(\psi, \alpha) - \sigma(\tilde{\psi}, \alpha)\|_1
\]

\[
+ CQ_1(\psi - \tilde{\psi}) \leq 5C\|\rho(t) - \tilde{\rho}(t)\|_1 + (5\sqrt{\alpha} + 1)C\alpha Q_1(\psi - \tilde{\psi})
\]

\[
\leq 5C\|\rho(t) - \tilde{\rho}(t)\|_1 + 8(5\sqrt{\alpha} + 1)C\|\psi - \tilde{\psi}\|_1,
\]

where we have used the fact that \( \|\sigma(\psi, \alpha) - \sigma(\tilde{\psi}, \alpha)\|_1 \leq \sqrt{\alpha}Q_1(\psi - \tilde{\psi}) \leq 8\sqrt{\alpha}\|\psi - \tilde{\psi}\|_1 \), which comes from both the properties of the invariant manifold and the inequalities of Lemma 2.1.

Then, the integral form of the initial value problem (4.16) is given by

\[
\psi^*(t; \psi, \sigma + \rho) = \psi + \alpha \int_0^t G(\psi^*(u; \psi, \sigma + \rho), \alpha) + \rho(u), \psi^*(u; \psi, \sigma + \rho), \alpha)du.
\]

Using this integral form and the above previously proven inequality we obtain

\[
\|\psi^*(t; \psi, \sigma + \rho) - \psi^*(t; \psi, \sigma + \tilde{\rho})\|_1
\]

\[
\leq \alpha \int_0^t \left[ 5C\|\rho(u) - \tilde{\rho}(u)\|_1 + 8(5\sqrt{\alpha} + 1)C\|\psi^*(u; \psi, \sigma + \rho) - \psi^*(u; \psi, \sigma + \tilde{\rho})\|_1 \right]du
\]

\[
\leq \alpha \int_0^t \left[ 5C\|\rho - \tilde{\rho}\|_Y + 8(5\sqrt{\alpha} + 1)C\|\psi^*(u; \psi, \sigma + \rho) - \psi^*(u; \psi, \sigma + \tilde{\rho})\|_1 \right]du
\]

\[
= 5\alpha\|\rho - \tilde{\rho}\|_Y t + 8(5\sqrt{\alpha} + 1)C\int_0^t \|\psi^*(u; \psi, \sigma + \rho) - \psi^*(u; \psi, \sigma + \tilde{\rho})\|_1du.
\]

From here the bound (4.17) is obtained by an application of Grönwall’s inequality, and follows as in the proof of (4.5b). Therefore, we omit these final steps due to their redundancy and complete the proof of the lemma.

Now, taking \( s = \sigma + \rho \), we use the differential equation (3.11) and the fact that \( \sigma \) is an invariant manifold of this differential equation to obtain the autonomous dynamical system governing the evolution of \( \rho \):

\[
\dot{\rho} = F(\psi^*(t; \psi, \sigma + \rho), \alpha) + \rho, \psi^*(t; \psi, \sigma + \rho), \alpha) - F(\psi^*(t; \psi, \sigma + \rho), \alpha), \psi^*(t; \psi, \sigma + \rho), \alpha), \quad (4.18)
\]

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which we use the definition of $F$ in (4.10) to write explicitly as

$$
\dot{\rho}_{i,j} = \alpha \sum_{i',j'} \left[ \rho_{i',j'} \cos(\tilde{\theta}_{i',j'}(\alpha) + \psi^*_{i',j'}(t; \psi, \sigma + \rho)) - \tilde{\theta}_{i,j}(\alpha) - \psi^*_2(t; \psi, \sigma + \rho)) - \rho_{i,j} \right] \\
+ (\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma + \rho), \alpha) + \rho_{i,j}) \lambda(\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma + \rho), \alpha) + \rho_{i,j}) \\
- (\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma + \rho), \alpha)) \lambda(\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma + \rho), \alpha)) \\
(4.19)
$$

for each $(i, j) \in \mathbb{Z}^2$. Note that this differential equation has a steady-state solution $\rho = 0$, representing the flow on the invariant manifold since $\rho$ is being used to capture deviations from this manifold.

We may further recast (4.18) using the function $\tilde{F}$ introduced in (4.10) so that

$$
\rho = a\lambda'(a) \rho + \tilde{F}(\sigma(\psi^*(t; \psi, \sigma + \rho), \alpha) + \rho(\psi^*(t; \psi, \sigma + \rho), \alpha), \psi^*(t; \psi, \sigma + \rho), \alpha),
$$

which in turn allows one to apply the variation of constants formula for the differential equation (4.18) with initial value $\rho(0) = \rho^0 \in \ell^1$ to obtain

$$
\rho(t) = e^{a\lambda'(a)t} \rho^0 + \int_0^t e^{a\lambda'(a)(t-u)} \left[ \tilde{F}(\sigma(\psi^*(u; \psi, \sigma + \rho)) + \rho(u), \psi^*(u; \psi, \sigma + \rho), \alpha) \\
- \tilde{F}(\sigma(\psi^*(u; \psi, \sigma + \rho)), \psi^*(u; \psi, \sigma + \rho)), \alpha) \right] du.
$$

Let us then define the mapping, denoted $T_2$, as

$$
T_2 \rho(t) := e^{a\lambda'(a)t} \rho^0 + \int_0^t e^{a\lambda'(a)(t-u)} \left[ \tilde{F}(\sigma(\psi^*(u; \psi, \sigma + \rho)) + \rho(u), \psi^*(u; \psi, \sigma + \rho), \alpha) \\
- \tilde{F}(\sigma(\psi^*(u; \psi, \sigma + \rho)), \psi^*(u; \psi, \sigma + \rho), \alpha) \right] du.
$$

(4.20)

so that fixed points of $T_2$ correspond to solutions of (4.18) with initial value $\rho(0) = \rho^0 \in \ell^1$.

**Lemma 4.7.** There exists $\delta_1, \alpha, \gamma_1 > 0$ such that for all $\alpha \in (0, \gamma_1)$, $\delta \in (0, \delta_1]$ and $\rho^0 \in \ell^1$ with $\|\rho^0\|_1 \leq \delta$ we have that $T_2 : Y(\delta) \rightarrow Y(\delta)$ is well-defined.

**Proof.** As in the proof of Lemma 4.3 we break the integrand up into smaller components to make things more manageable, then put them back together at the end. To begin, we use the definition of $\tilde{F}$ along with (4.19) to note that

$$
\tilde{F}(\sigma(\psi^*(t; \psi, \sigma + \rho)) + \rho(t), \psi^*(t; \psi, \sigma + \rho), \alpha) - \tilde{F}(\sigma(\psi^*(t; \psi, \sigma + \rho)), \psi^*(t; \psi, \sigma + \rho), \alpha) \\
= \alpha \sum_{i',j'} \left[ \rho_{i',j'} \cos(\tilde{\theta}_{i',j'}(\alpha) + \psi^*_{i',j'}(t; \psi, \sigma + \rho) - \tilde{\theta}_{i,j}(\alpha) - \psi^*_2(t; \psi, \sigma + \rho)) - \rho_{i,j} \right] \\
+ (\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, f + \rho), \alpha) + \rho_{i,j}) \lambda(\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma + \rho), \alpha) + \rho_{i,j}) \\
- (\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, f + \rho), \alpha)) \lambda(\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma + \rho), \alpha)) - a\lambda'(a) \rho,
$$

(4.21)

where the only change from (4.19) is that addition of the $-a\lambda'(a)\rho$ term at the end.
Now,

\[ |\rho_{i,j}^* \cos(\tilde{\theta}_{i,j}(\alpha) + \psi_{i,j}^*(t; \psi, \sigma + \rho) - \tilde{\theta}_{i,j}(\alpha) - \psi_{i,j}^*(t; \psi, \sigma + \rho))| \leq |\rho_{i,j}^*|, \]

for all \((i, j) \in \mathbb{Z}^2\). This therefore gives that

\[ \alpha \left| \sum_{i,j} \rho_{i,j}^* \cos(\tilde{\theta}_{i,j}(\alpha) + \psi_{i,j}^*(t; \psi, \sigma + \rho) - \tilde{\theta}_{i,j}(\alpha) - \psi_{i,j}^*(t; \psi, \sigma + \rho)) - \rho_{i,j} \right| \leq 4\alpha |\rho_{i,j}| + \alpha \sum_{i,j} |\rho_{i,j}^*|. \]

Furthermore, the remaining parts of (4.21) can be compactly written as the function

\[(x_{i,j} + \rho_{i,j})\lambda(x_{i,j} + \rho_{i,j}) - x_{i,j} \lambda(x_{i,j}) - a\lambda'(a)\rho_{i,j},\]

where \(x_{i,j} = \tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma + \rho), \alpha)\) for notational convenience. Then, since the terms \(\tilde{r}_{i,j}(\alpha) + \sigma_{i,j}(\psi^*(t; \psi, \sigma + \rho), \alpha)\) are uniformly bounded, a nearly identical argument to that employed in (4.12) results in the existence of a constant \(C'_\delta > 0\) such that

\[ |(x_{i,j} + \rho_{i,j})\lambda(x_{i,j} + \rho_{i,j}) - x_{i,j} \lambda(x_{i,j}) - a\lambda'(a)\rho_{i,j}| \leq C'_\delta(\alpha + \delta)|\rho_{i,j}|, \]

for every \(\delta \in [0, 1]\) and \(\alpha \leq \min\{\alpha^*, a\},\) uniformly in \((i, j) \in \mathbb{Z}^2\).

Putting this all together then gives that

\[
\begin{align*}
    \| \tilde{F}(\sigma(\psi^*(t; \psi, \sigma + \rho)) + \rho, \psi^*(t; \psi, \sigma + \rho), \alpha) - \tilde{F}(\sigma(\psi^*(t; \psi, \sigma + \rho)), \psi^*(t; \psi, \sigma + \rho), \alpha) \|_1 \\
    \leq 4\alpha \sum_{i,j} |\rho_{i,j}| + \alpha \sum_{i,j} \sum_{i',j'} |\rho_{i',j'}^*| + C'_\delta(\alpha + \delta) \sum_{i,j} |\rho_{i,j}|
    = 4\alpha \|\rho(t)\|_1 + 4\alpha \|\rho\|_1 + C'_\delta(\alpha + \delta)\|\rho\|_1
    = (8\alpha + C'_\delta\alpha + C'_\delta\delta)\|\rho\|_1.
\end{align*}
\]

Then, this therefore implies that for any \(\rho \in Y(\delta)\) we have

\[
\begin{align*}
    \|T_2\rho(t)\|_1 &\leq e^{a\lambda'(a)t}\|\rho^0\|_1 + (8\alpha + C'_\delta\alpha + C'_\delta\delta) \int_0^t e^{a\lambda'(a)(t-u)}\|\rho(u)\|_1 du \\
    &\leq e^{a\lambda'(a)t}\|\rho^0\|_1 + 2\delta(8\alpha + C'_\delta\alpha + C'_\delta\delta) \int_0^t e^{a\lambda'(a)(t-u)} e^{a\lambda'(a)u} du \\
    &= e^{a\lambda'(a)t}\|\rho^0\|_1 + 4\delta(8\alpha + C'_\delta\alpha + C'_\delta\delta) \left( e^{a\lambda'(a)\frac{t}{2}} - e^{-a\lambda'(a)\frac{t}{2}} \right) \\
    &\leq e^{a\lambda'(a)t}\|\rho^0\|_1 + 4\delta(8\alpha + C'_\delta\alpha + C'_\delta\delta) e^{a\lambda'(a)\frac{t}{2}}.
\end{align*}
\]

Then, taking \(\|\rho^0\|_1 \leq \delta\), we can use the fact that \(e^{a\lambda'(a)t} \leq e^{\frac{a\lambda'(a)t}{2}}\) for all \(t \geq 0\) to see that for all \(\delta \in [0, 1]\) we have

\[
\|T_2\rho(t)\|_1 \leq \delta \left( 1 + \frac{\alpha(8 + C'_\delta) + 4\delta C'_\delta}{|a\lambda'(a)|} \right) e^{a\lambda'(a)\frac{t}{2}},
\]

and therefore one sees that upon taking

\[ \alpha_{Y,1} := \min\left\{ \alpha^*, \frac{a}{4} \left| a\lambda'(a) \right| \right\} \]

\[ \frac{16 + 2C'_\delta}{\lambda'(a)} \]

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and
\[ \delta \leq \min \left\{ \frac{|a\lambda'(a)|}{8C_\lambda}, 1 \right\}, \]
gives that
\[ ||T_2\rho(t)||_1 \leq 2\delta e^{-\frac{a\lambda'(a)}{2}}, \]
for all \( \alpha \in [0, \alpha_Y, 1] \) and \( \delta > 0 \) chosen appropriately small. Our choices of \( \alpha \) and \( \delta \) therefore show that \( T_2 \) maps elements of \( Y(\delta) \) back into \( Y(\delta) \), completing the proof.

**Lemma 4.8.** There exists \( \delta_2, \alpha_Y, 2 > 0 \) such that for all \( \alpha \in [0, \alpha_Y, 2] \), \( \delta \in (0, \delta_2] \) and \( \rho^0 \in \ell^1 \) with \( \|\rho^0\|_1 \leq \delta \) we have that \( T_2 \) is a contraction on \( Y(\delta) \) with contraction constant at most \( \frac{1}{2} \).

**Proof.** The proof uses nearly identical manipulations to those employed in the proof of Lemma 4.7 and therefore we merely focus on those aspects that differentiate it. First, to obtain Lipschitz properties of the integrand
\[
\tilde{F}(\sigma(\psi^*(u; \psi, \sigma + \rho))) + \rho(u), \psi^*(u; \psi, \sigma + \rho), \alpha) - \tilde{F}(\sigma(\psi^*(u; \psi, \sigma + \rho))) = F^*(u; \psi, \sigma + \rho), \alpha) , \] (4.22)
with respect to \( \rho \) much of the manipulations follow in a similar way to that of Lemma 4.4 except the term
\[
|\rho_{i,j'} \cos(\tilde{\theta}_{i,j'}(\alpha) + \psi_{i,j'}^*(t; \psi, \sigma + \rho)) - \tilde{\theta}_{i,j}(\alpha) - \psi_{i,j}(t; \psi, \sigma + \rho))| - \tilde{\theta}_{i,j'}(\alpha) - \psi_{i,j}(t; \psi, \sigma + \rho)),
\]
which requires one to utilize the results of Lemma 4.6. That is, for \( C_3 > 0 \), the constant guaranteed by Lemma 4.6, we obtain the bound
\[
\sum_{i,j} \sum_{i', j'} |\rho_{i,j'} \cos(\tilde{\theta}_{i,j'}(\alpha) + \psi_{i,j'}^*(t; \psi, \sigma + \rho)) - \tilde{\theta}_{i,j}(\alpha) - \psi_{i,j}(t; \psi, \sigma + \rho))| \leq 8e^{\alpha C_4 t} \rho_0 \|Y\| + |\rho_{i,j'} - \rho_{i,j'}|.
\]
for all \( t \geq 0 \). Then, taking the sum over \( (i,j) \in \mathbb{Z}^2 \) gives
\[
\sum_{i,j} \sum_{i', j'} |\rho_{i,j'} \cos(\tilde{\theta}_{i,j'}(\alpha) + \psi_{i,j'}^*(t; \psi, \sigma + \rho)) - \tilde{\theta}_{i,j}(\alpha) - \psi_{i,j}(t; \psi, \sigma + \rho))| \leq 8e^{\alpha C_4 t} \rho_0 \|Y\| + |\rho_{i,j'} - \rho_{i,j'}|\leq 4(8e^{\alpha C_4 t} \rho(t) \|1\| + 1) \rho_0 \|Y\| \leq 4(16e^{\alpha C_4 t} \frac{\lambda'(a)}{2} + 1) ||\rho - \rho||_Y,\]
where we have applied the fact that \( \rho \in Y(\delta) \) implies \( \|\rho(t)\|_1 \leq 2\delta e^{-\frac{\alpha\lambda'(a)}{2}t} \). Then, taking \( \alpha \) appropriately small will guarantee that
\[
\alpha C_3 + \frac{\alpha\lambda'(a)}{2} < 0,
\]
and hence the bound \( 4(16\delta e^{(\alpha C_3 + \frac{\alpha\lambda'(a)}{2})t} + 1) \) is bounded uniformly in \( t \geq 0 \).

From here much of the work is the same as in Lemmas 4.4 and 4.7. Upon obtaining Lipschitz bounds on the integrand \( (4.22) \), we simply use the fact that the integral
\[
\int_0^t e^{\alpha\lambda'(a)(t-u)}\,du
\]
is uniformly bounded in \( t \) (since \( \alpha\lambda'(a) < 0 \)), showing that \( T_2 \) can be made a contraction via appropriately small choices of \( \alpha, \delta > 0 \). Therefore we state this result without a full proof since it follows in a similar way to much of the work that has been done in this subsection and the one previous to it.

Lemmas 4.7 and 4.8 combine to show that if we choose \( \|\rho^0\|_1 \) sufficiently small, we have that the solution \( \rho(t) \) to \( (4.18) \) converges exponentially to 0. This means that for the original system \( (3.11) \), if we start sufficiently close to the invariant manifold, we will converge exponentially fast to the invariant manifold. This therefore completes the proof of Theorem 3.1.

5 Stability on the Invariant Manifold

As previously noted, system \( (3.11) \) is a fast-slow system which is singular at \( \alpha = 0 \). Now that we have analyzed the fast component, \( s \), we wish to inspect the slow component, \( \psi \). We begin by using Theorem 3.1 to write the fast variable as \( s = \sigma(\psi, \alpha) + \rho \). Furthermore, the work of the previous section implies that the specific evolution of \( \rho(t) \) depends on the evolution of the phase variable, \( \psi(t) \). Coupling the global Lipschitz properties of the function \( G \) with respect to \( \psi \) with the results of Lemmas 4.2 and 4.6 implies that for all \( \psi^0 \in \ell^1 \) a global solution \( \psi(t) \) exists and belongs to \( \ell^1 \) for all \( t \geq 0 \). Hence, for any fixed \( \psi^0 \in \ell^1 \) we consider \( \rho(t) = \rho(t; \psi^0) \) to be the solution of \( (4.18) \) with \( \rho(0) = \rho^0 \) such that \( \|\rho^0\|_1 \leq \delta \). Then, the results of Theorem 3.1 imply that \( \|\rho(t)\|_1 \leq 2\delta e^{-\beta t} \), for some \( \beta > 0 \), regardless of our choice of \( \psi^0 \in \ell^1 \).

Having now solved the fast variable equation, we therefore introduce the slow-time variable \( \tau := \alpha t \), for \( \alpha > 0 \). This results in the system
\[
\frac{d}{d\tau} \psi = G(\sigma(\psi, \alpha) + \rho(\alpha^{-1}\tau), \psi(\tau), \alpha),
\] (5.1)
along with an initial condition \( \psi(0) = \psi^0 \in \ell^1 \), so that \( \rho(t) \) is well-defined. We note that \( \|\rho(\alpha^{-1}\tau)\|_1 \leq 2\delta e^{-\frac{\beta}{\alpha} \tau} \), for \( \alpha > 0 \) so long as \( \|\rho^0\|_1 \leq \delta \), for \( \delta > 0 \) taken sufficiently small. It is therefore through system \( (5.1) \) which we plan to investigate the stability of \( \psi = 0 \) in this section. As in the previous section, we will assume that Hypotheses \( \{1\} \) hold throughout, but do not explicitly say this in the statement of all results in this section.
5.1 Integral Bounds

In this section we provide two important integral bounds which will be used throughout the proof of the stability on the invariant manifold. The first of which is a restatement of a result which can be found in [7] and therefore is stated without proof, whereas the second will be stated with proof since an appropriate reference could not be found.

Lemma 5.1 (§3, Lemma 3.2). Let \( \gamma_1, \gamma_2 \) be positive real numbers. If \( \gamma_1, \gamma_2 \neq 1 \) or if \( \gamma_1 = 1 < \gamma_2 \) then there exists a \( C_{\gamma_1, \gamma_2} > 0 \) (continuously depending on \( \gamma_1 \) and \( \gamma_2 \)) such that

\[
\int_0^\tau (1 + \tau - u)^{-\gamma_1} (1 + u)^{-\gamma_2} du \leq C_{\gamma_1, \gamma_2} (1 + \tau)^{-\min\{\gamma_1 + \gamma_2 - 1, \gamma_1 \gamma_2\}},
\]

for all \( \tau \geq 0 \).

Lemma 5.2. If \( \beta, \gamma \) are positive constants, there exists a constant \( C_{\gamma, \beta} > 0 \) (continuously depending on \( \beta \) and \( \gamma \)) such that

\[
\int_0^\tau (1 + \tau - u)^{-\gamma} e^{-\frac{\beta}{\tau} u} du \leq \alpha C_{\gamma, \beta} (1 + \tau)^{-\gamma},
\]

for all \( \tau \geq 0 \) and \( \alpha \in (0, 1] \).

Proof. We begin by integrating by parts to obtain

\[
\int_0^\tau (1 + \tau - u)^{-\gamma} e^{-\frac{\beta}{\tau} u} du = \frac{\alpha}{\beta} (1 + \tau)^{-\gamma} - \frac{\alpha}{\beta} e^{-\frac{\beta}{\tau} \tau} + \frac{\alpha\gamma}{\beta} \int_0^\tau (1 + \tau - u)^{-\gamma} e^{-\frac{\beta}{\tau} u} du.
\]

Then, the rightmost integral can be bounded as

\[
\int_0^\tau (1 + \tau - u)^{-\gamma} e^{-\frac{\beta}{\tau} u} du \leq (1 + \frac{\tau}{2})^{-\gamma} \int_0^\frac{\tau}{2} e^{-\frac{\beta}{\tau} u} du + e^{-\frac{\beta}{\tau} \tau} \int_\frac{\tau}{2}^\tau (1 + \tau - u)^{-\gamma} e^{-\frac{\beta}{\tau} u} du
\]

\[
\leq \frac{\alpha}{\beta} (1 + \frac{\tau}{2})^{-\gamma} \left[ 1 - e^{-\frac{\beta}{\tau} \tau} \right] + \frac{1}{\gamma} e^{-\frac{\beta}{\tau} \tau} \left[ 1 - (1 + \frac{\tau}{2})^{-\gamma} \right]
\]

\[
\leq \frac{\alpha}{\beta} (1 + \frac{\tau}{2})^{-\gamma} + \frac{1}{\gamma} e^{-\frac{\beta}{\tau} \tau}.
\]

Putting this bound back into (5.3) we arrive at

\[
\int_0^\tau (1 + \tau - u)^{-\gamma} e^{-\frac{\beta}{\tau} u} du \leq \frac{\alpha}{\beta} (1 + \tau)^{-\gamma} - \frac{\alpha}{\beta} e^{-\frac{\beta}{\tau} \tau} + \frac{\alpha^2 \gamma}{\beta^2} (1 + \frac{\tau}{2})^{-\gamma} + \frac{\alpha}{\beta} e^{-\frac{\beta}{\tau} \tau}
\]

\[
= \frac{\alpha}{\beta} (1 + \tau)^{-\gamma} + \frac{\alpha^2 \gamma}{\beta^2} (1 + \frac{\tau}{2})^{-\gamma} + \frac{\alpha}{\beta} \left[ e^{-\frac{\beta}{\tau} \tau} - e^{-\frac{\beta}{\tau} \tau} \right]
\]

\[
\leq \frac{\alpha}{\beta} (1 + \tau)^{-\gamma} + \frac{\alpha^2 \gamma}{\beta^2} (1 + \frac{\tau}{2})^{-\gamma}.
\]

Therefore, when \( 0 < \alpha \leq 1 \) we can find a \( C > 0 \), independent of \( \alpha \) and \( \tau \), so that

\[
\frac{\alpha^2 \gamma}{\beta^2} (1 + \frac{\tau}{2})^{-\gamma} \leq \frac{\alpha^2 \gamma}{\beta^2} C (1 + \tau)^{-\gamma}
\]
for all $\tau \geq 0$, thus completing the proof.

**Remark 3.** We note that the constant $C_{\gamma_1, \gamma_2}$ used in the statement of Lemma 5.1 depends continuously on $\gamma_1, \gamma_2$. Hence, if Lemma 5.1 is applied for a range of $\gamma_1, \gamma_2$ we are able to uniformly bound $C_{\gamma_1, \gamma_2}$ provided that the full range of $\gamma_1, \gamma_2$ belong to a compact set. This will be the case throughout the following sections. The same statement applies to the constant $C_{\gamma, \beta}$ from Lemma 5.2.

### 5.2 Semigroup Decay

Let us consider the linear operator $\tilde{L}_\alpha$, parametrized by $\alpha$, acting upon the sequences $x = \{x_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ by

$$[	ilde{L}_\alpha x]_{i,j} = \sum_{i',j'} \tilde{r}_{i',j'}(\alpha) \cos(\tilde{\theta}_{i',j'}(\alpha) - \tilde{\theta}_{i,j}(\alpha))(x_{i',j'} - x_{i,j})$$

for all $(i, j) \in \mathbb{Z}^2$. The following proposition details how the decay properties of the semigroup $e^{L_\alpha t}$, where $L_\alpha$ is defined in (3.13), can be extended to give decay properties of the semigroup $e^{\tilde{L}_\alpha t}$.

**Proposition 5.3.** There exists a constant $\tilde{C}_L > 0$ such that for all $x_0 \in \ell^1$, $\alpha \in [0, \alpha^*]$, and $t \geq 0$ we have

$$\|e^{L_\alpha t}x_0\|_p \leq \tilde{C}_L (1 + t)^{-1 + \frac{1}{p}} \|x_0\|_1,$$

$$Q_p(e^{L_\alpha t}x_0) \leq \tilde{C}_L (1 + t)^{-\min(1 - \frac{1}{p}, \beta + \gamma)} \|x_0\|_1,$$

where $e^{\tilde{L}_\alpha t}$ is the semi-group with infinitesimal generator given by $\tilde{L}_\alpha$.

We see that the $\ell^p$ norm decay of the semigroup $e^{L_\alpha}$ is asymptotically equivalent to those of the semigroup $e^{\tilde{L}_\alpha}$ guaranteed by Hypothesis 3 but the $Q_p$ semi-norm decay has a slight adjustment for $p \geq q^*$. In Lemma 5.5 we will see that this is a consequence of Hypothesis 4 and moreover, if $p^* = 1$ the decay with respect to the $Q_p$ semi-norms are asymptotically equivalent to those of the semigroup $e^{L_\alpha}$.

To emphasize where these differences in decay rates come from, the proof of Proposition 5.3 is broken down into the following series of results.

**Lemma 5.4.** There exists a constant $C_{q^*} > 0$ such that for all $x \in \ell^1$ and $\alpha \in [0, \alpha^*]$ we have

$$\|(\tilde{L}_\alpha - L_\alpha)x\|_1 \leq C_{q^*} Q_{q^*}(x),$$

where $q^*$ is the Hölder conjugate of $p^*$ from Hypothesis 4.

**Proof.** Using the definitions of $\tilde{L}_\alpha$ and $L_\alpha$, we find that for any $x = \{x_{i,j}\}_{(i,j) \in \mathbb{Z}^2} \in \ell^1$ we have

$$[(\tilde{L}_\alpha - L_\alpha)x]_{i,j} = \sum_{i',j'} \tilde{r}_{i',j'}(\alpha) - \tilde{\theta}_{i',j'}(\alpha))(x_{i',j'} - x_{i,j}),$$

for all $(i, j) \in \mathbb{Z}^2$ and $\alpha \in [0, \alpha^*]$. Then, since $\frac{q}{2} \leq \tilde{r}_{i,j}(\alpha) \leq \frac{3q}{2}$ for all $(i, j)$ and $\alpha \in [0, \alpha^*]$, there exists a uniform Lipschitz constants, $M_r > 0$, such that for all $(i, j) \in \mathbb{Z}^2$ and $\alpha \in [0, \alpha^*]$ we have

$$\|[(\tilde{L}_\alpha - L_\alpha)x]_{i,j}\| \leq M_r \sum_{i',j'} \left( |\tilde{r}_{i',j'}(\alpha) - a| + |\tilde{\theta}_{i,j}(\alpha) - a| \right) |x_{i',j'} - x_{i,j}|,$$
since $|\cos(\hat{\theta}_{i,j'}(\alpha) - \hat{\theta}_{i,j}(\alpha))| \leq 1$ for all $(i, j) \in \mathbb{Z}^2$. Then, taking the sum over all $(i, j)$ we arrive at
\[
\| (\tilde{L}_\alpha - L_\alpha) x \|_1 = \sum_{(i, j) \in \mathbb{Z}^2} |[(\tilde{L}_\alpha - L_\alpha) x]_{i,j} |
\]
\[
\leq M_r \sum_{(i, j) \in \mathbb{Z}^2} \sum_{i', j'} \left( |\tilde{\varphi}_{i', j'}(\alpha) - a| + |\tilde{\varphi}_{i,j}(\alpha) - a| \right) |x_{i', j'} - x_{i,j}|,
\]
\[
\leq M_r \sum_{(i, j) \in \mathbb{Z}^2} \left[ \left( \sum_{i', j'} |\tilde{\varphi}_{i,j}(\alpha) - a|^{p_r} \right)^{\frac{1}{p_r}} + \left( \sum_{i', j'} |\tilde{\varphi}_{i', j'}(\alpha) - a|^{p_r} \right)^{\frac{1}{p_r}} \right] \left( \sum_{i', j'} |x_{i', j'} - x_{i,j}| \right),
\]
\[
\leq M_r \left( \sum_{(i, j) \in \mathbb{Z}^2} \sum_{i', j'} |\tilde{\varphi}_{i,j}(\alpha) - a|^{p_r} \right)^{\frac{1}{p_r}} Q_{q^*}(x).
\]
where we have applied Hölder’s inequality twice, once for the nearest-neighbour summation and once for the full summation over the full set of indices. Finally, from Hypothesis 4 we have that
\[
Q_{q^*} \leq C_{sol}^r < \infty,
\]
completing the proof. \qed

Now, if $x(t)$ is a solution to the differential equation
\[
\dot{x} = \tilde{L}_\alpha x
\]
with initial condition $x(0) = x_0 \in \ell^1$, then this solution $x(t)$ is such that $x(t) = e^{L_\alpha t} x_0$, and hence understanding the decay of solutions to (5.6) leads to the proof of Proposition 5.3. Trivially we have that
\[
\dot{x}(t) = L_\alpha x(t) + (\tilde{L}_\alpha - L_\alpha) x(t),
\]
and using the variation of constants formula we obtain the equivalent integral form of the ordinary differential equation (5.6), given as
\[
x(t) = e^{L_\alpha t} x_0 + \int_0^t e^{L_\alpha (t-s)} (\tilde{L}_\alpha - L_\alpha) x(s) ds,
\]
where we again recall that $e^{L_\alpha t}$ is the semigroup with infinitesimal generator $L_\alpha$ with the decay properties given in Hypothesis 3. We now use the integral form (5.7) to prove Proposition 5.3.

**Lemma 5.5.** There exists a constant $C_Q > 0$ such that for every $x_0 \in \ell^1$ and $\alpha \in [0, \alpha^*]$, the solution $x(t) = e^{L_\alpha t} x_0$ to (5.6) with initial condition $x(0) = x_0$ satisfies
\[
Q_{q^*}(x(t)) \leq C_Q (1 + t)^{-\frac{1}{p_r} - \eta} \|x_0\|_1,
\]
for all $t \geq 0$. 

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Proof. Through straightforward manipulations of the integral form \([5.7]\) one finds that

\[
Q_{q^*}(x(t)) \leq Q_{q^*}(e^{\tilde{L}_\alpha t}x_0) + \int_0^t Q_{q^*}(e^{\tilde{L}_\alpha (t-s)}(\tilde{L}_\alpha - L_\alpha)x(s))ds.
\]

Then, using Hypothesis \([5]\) we obtain

\[
Q_{q^*}(x(t)) \leq C_L(1 + t)^{-1 + 1/p^* - \eta}\|x_0\|_1 + C_L \int_0^t (1 + t - s)^{-1 + 1/p^* - \eta}\|\tilde{L}_\alpha - L_\alpha\|_1ds
\]

\[
\leq C_L(1 + t)^{-1 + 1/p^* - \eta}\|x_0\|_1 + C_L C_{q^*} \int_0^t (1 + t - s)^{-1 + 1/p^* - \eta}e^{C_L C_{q^*} f^*_r(1+t-r)^{-1 + 1/p^* - \eta}dr}ds.
\]

where we have applied the results of Lemma \([5.3]\). We now apply Gronwall’s inequality to see that

\[
Q_{q^*}(x(t)) \leq C_{\text{exp}} > 0
\]

Then, combining this bound with the result of Lemma \([5.1]\) we find that

\[
\int_0^t (1 + t - s)^{-1 + 1/p^* - \eta}(1 + s)^{-1 + 1/p^* - \eta}e^{C_L C_{q^*} f^*_r(1+t-r)^{-1 + 1/p^* - \eta}dr}ds \leq C_{\text{exp}} C_{1 - 1/p^* + \eta, 1 - 1/p^* + \eta}(1 + t)^{-1 + 1/p^* - \eta}.
\]

Putting this all together therefore gives

\[
Q_{q^*}(x(t)) \leq C_L(1 + C_{\text{exp}} C^2_L C_{q^*} C_{1 - 1/p^* + \eta, 1 - 1/p^* + \eta})(1 + t)^{-1/q^* - \eta}\|x_0\|_1,
\]

which upon writing

\[
-1 + \frac{1}{q^*} - \eta = -\frac{1}{p^*} - \eta
\]

completes the proof.

The further decay properties stated in Proposition \([5.3]\) now follow in a straightforward way from the results of Lemma \([5.3]\).

Proof of Proposition \([5.3]\). We begin as in the proof of Lemma \([5.5]\) by applying elementary manipulations to the integral form \([5.7]\) in order to find that

\[
\|x(t)\|_p \leq \|e^{\tilde{L}_\alpha t}x_0\|_p + \int_0^t \|e^{\tilde{L}_\alpha (t-s)}(\tilde{L}_\alpha - L_\alpha)x(s)\|_pds,
\]
for every \( p \in [1, \infty] \). Then, using Hypothesis 3 and the results of Lemma 5.5, we obtain
\[
\|x(t)\|_p \leq C_L (1 + t)^{-1 + \frac{1}{p}} \|x_0\|_1 + C_L \int_0^t (1 + t - s)^{-1 + \frac{1}{p}} \|\tilde{L}_\alpha - L_\alpha\| x(s) \|_1 ds
\]
\[
\leq C_L (1 + t)^{-1 + \frac{1}{p}} \|x_0\|_1 + C_L C_q \int_0^t (1 + t - s)^{-1 + \frac{1}{p}} Q_q(x(s)) ds
\]
\[
\leq C_L (1 + t)^{-1 + \frac{1}{p}} \|x_0\|_1 + C_q C_L C_q^* \|x_0\|_1 \int_0^t (1 + t - s)^{-1 + \frac{1}{p}} (1 + s)^{-\frac{1}{p} - \eta} ds.
\]
From Lemma 5.1 we find that there exists \( C_1 \frac{1}{p^*} + \eta > 0 \) (continuously depending on \( 1 - \frac{1}{p} \)) such that
\[
\int_0^t (1 + t - s)^{-1 + \frac{1}{p}} (1 + s)^{-\frac{1}{p} - \eta} ds \leq C_1 \frac{1}{p^*} + \eta (1 + t)^{-1 + \frac{1}{p}},
\]
so that
\[
\|x(t)\|_p \leq C_L (1 + C_q C_q^*) (1 + t)^{-1 + \frac{1}{p}} \|x_0\|_1,
\]
and using the fact that \( 1 - \frac{1}{p} \in [0, 1] \) for \( p \in [1, \infty] \), we may bound \( C_1 \frac{1}{p^*} + \eta \) uniformly in \( p \), providing the first decay bound of (5.5). The second decay bound of (5.5) follows through a nearly identical manipulation, eventually making use of Lemma 5.1 by bounding
\[
\int_0^t (1 + t - s)^{-1 + \frac{1}{p} - \eta} (1 + s)^{-\frac{1}{p} - \eta} ds \leq C_1 \frac{1}{p^*} + \eta (1 + t)^{-\min(1 - \frac{1}{p^*}, \frac{1}{p} + \eta)},
\]
which then leads to the the second decay bound of (5.5). This completes the proof.

5.3 Nonlinear Estimates

Here we provide Lipschitz estimates on the function \( G(s, \psi, \alpha) \) in terms of \( (s, \psi) \in \ell^1 \times \ell^1 \), uniformly in \( \alpha \in [0, \alpha^*] \). Recall that \( G(0, 0, \alpha) = 0 \) for all \( \alpha \in [0, \alpha^*] \). We introduce the functions \( G^{(1)}, G^{(2)}, \) and \( G^{(3)} \) given by
\[
G_{i,j}^{(1)}(s, \psi, \alpha) = \sum_{i', j'} \left( \frac{\bar{r}_{i', j'}(\alpha) + s_{i', j'}}{\bar{r}_{i,j}(\alpha) + s_{i,j}} - \frac{\bar{r}_{i', j'}(\alpha)}{\bar{r}_{i,j}(\alpha)} \right) \sin(\bar{\theta}_{i', j'}(\alpha) + \psi_{i', j'} - \bar{\theta}_{i,j}(\alpha) - \psi_{i,j}),
\]
\[
G_{i,j}^{(2)}(\psi, \alpha) = \sum_{i', j'} \left[ \frac{\bar{r}_{i', j'}(\alpha)}{\bar{r}_{i,j}(\alpha)} \left( \sin(\bar{\theta}_{i', j'}(\alpha) + \psi_{i', j'} - \bar{\theta}_{i,j}(\alpha) - \psi_{i,j}) - \sin(\bar{\theta}_{i', j'}(\alpha) - \bar{\theta}_{i,j}(\alpha)) \right)
\]
\[
- \cos(\bar{\theta}_{i', j'}(\alpha) - \bar{\theta}_{i,j}(\alpha)) (\psi_{i', j'} - \psi_{i,j}) \right],
\]
\[
G_{i,j}^{(3)}(s, \alpha) = \omega_1 (\bar{r}_{i,j}(\alpha) + s_{i,j}, \alpha) - \omega_1 (\bar{r}_{i,j}(\alpha), \alpha),
\]
so that
\[
G(s, \psi, \alpha) = G(s, \psi, \alpha) - G(0, 0, \alpha) = \tilde{L}_\alpha \psi + G_{i,j}^{(1)}(s, \psi, \alpha) + G_{i,j}^{(2)}(\psi, \alpha) + G_{i,j}^{(3)}(s, \alpha).
\]
We now obtain Lipschitz estimates on the functions \( G^{(1)}, G^{(2)}, \) and \( G^{(3)} \).
Lemma 5.6. There exists a constant $C_{G,1} > 0$ such that for all $\psi \in \ell^1$, $\alpha \in [0, \alpha^*]$, and $s \in \ell^1$ with $\|s\|_\infty \leq \frac{3a}{8}$ we have

$$\|G^{(1)}(s, \psi, \alpha)\|_1 \leq C_{G,1}(\|s\|_{q^*} + \|s\|_2Q_2(\psi)).$$

Proof. We remark that the restriction $\|s\|_\infty \leq \frac{3a}{8}$ guarantees that the terms $\tilde{r}_{i,j}(\alpha) + s_{i,j}$ are uniformly bounded in absolute value away from zero. Hence, there exists a Lipschitz constant $K_1 > 0$ such that

$$\frac{|\tilde{r}_{i',j'}(\alpha) + s_{i',j'} - \tilde{r}_{i,j}(\alpha) + s_{i,j}|}{\tilde{r}_{i,j}(\alpha) + s_{i,j}} \leq K_1(|s_{i,j}| + |s_{i',j'}|),$$

for all $\alpha \in [0, \alpha^*]$ and $(i, j) \in \mathbb{Z}^2$. Similarly, we get the following estimate

$$|\sin(\tilde{\theta}_{i',j'}(\alpha) + \psi_{i',j'} - \tilde{\theta}_{i,j}(\alpha) - \psi_{i,j})| \leq |\sin(\tilde{\theta}_{i',j'}(\alpha) + \psi_{i',j'} - \tilde{\theta}_{i,j}(\alpha) - \psi_{i,j})| + |\sin(\tilde{\theta}_{i',j'}(\alpha) - \tilde{\theta}_{i,j}(\alpha))|$$

for all $(i, j)$ since sine is globally Lipschitz with Lipschitz constant one.

Putting this all together gives

$$\|G^{(1)}(s, \psi, \alpha)\|_1 \leq \sum_{(i,j) \in \mathbb{Z}^2} \sum_{i',j'} \left| \frac{\tilde{r}_{i',j'}(\alpha) + s_{i',j'} - \tilde{r}_{i,j}(\alpha)}{\tilde{r}_{i,j}(\alpha) + s_{i,j}} \right| |\sin(\tilde{\theta}_{i',j'}(\alpha) + \psi_{i',j'} - \tilde{\theta}_{i,j}(\alpha) - \psi_{i,j})|$$

$$\leq K_1 \sum_{(i,j) \in \mathbb{Z}^2} \sum_{i',j'} \left( |s_{i,j}| + |s_{i',j'}| \right) \left( |\psi_{i',j'} - \psi_{i,j}| + |\sin(\tilde{\theta}_{i',j'}(\alpha) - \tilde{\theta}_{i,j}(\alpha))| \right)$$

$$\leq K_1 \sum_{(i,j) \in \mathbb{Z}^2} \sum_{i',j'} \left( |s_{i,j}| + |s_{i',j'}| \right) \left( |\psi_{i',j'} - \psi_{i,j}| + |\sin(\tilde{\theta}_{i',j'}(\alpha) - \tilde{\theta}_{i,j}(\alpha))| \right)$$

(5.9)

uniformly in $\alpha \in [0, \alpha^*]$. Now, through a simple application of Hölder’s inequality one obtains

$$\sum_{(i,j) \in \mathbb{Z}^2} \sum_{i',j'} \left( |s_{i,j}| + |s_{i',j'}| \right) \left( |\sin(\tilde{\theta}_{i',j'}(\alpha) - \tilde{\theta}_{i,j}(\alpha))| \right) \leq 8\|s\|_2Q_2(\psi),$$

(5.10)

since each element has exactly four nearest-neighbours. Then, using Hölder’s inequality twice we find that

$$\sum_{(i,j) \in \mathbb{Z}^2} \sum_{i',j'} \left( |s_{i,j}| + |s_{i',j'}| \right) \left( |\sin(\tilde{\theta}_{i',j'}(\alpha) - \tilde{\theta}_{i,j}(\alpha))| \right)$$

$$\leq \sum_{(i,j) \in \mathbb{Z}^2} \left( \sum_{i',j'} \left( |s_{i,j}| + |s_{i',j'}| \right) \right)^{\frac{q}{q'}} \left( \sum_{i',j'} \left( |\sin(\tilde{\theta}_{i',j'}(\alpha) - \tilde{\theta}_{i,j}(\alpha))| \right) \right)^{\frac{q'}{q}}$$

$$\leq \left( \sum_{(i,j) \in \mathbb{Z}^2} \sum_{i',j'} \left( |s_{i,j}| + |s_{i',j'}| \right) \right)^{\frac{q}{q'}} \left( \sum_{(i,j) \in \mathbb{Z}^2} \sum_{i',j'} \left( |\sin(\tilde{\theta}_{i',j'}(\alpha) - \tilde{\theta}_{i,j}(\alpha))| \right) \right)^{\frac{q'}{q}}$$

(5.11)

$$\leq 8C_{40a}^{\frac{q}{q'}} \|s\|_{q^*},$$

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where $C_{\text{sol}} > 0$ is the constant guaranteed by Hypothesis 4 and we have again used the fact that each element has exactly four nearest-neighbours. Therefore, the bounds (5.10) and (5.11) together with (5.9) give the bound stated in the lemma.

**Lemma 5.7.** There exists a constant $C_{G,2} > 0$ such that for all $\psi \in \ell^1$ and $\alpha \in [0, \alpha^*]$ we have

$$\|G^{(2)}(\psi, \alpha)\|_1 \leq C_{G,2}Q_2^2(\psi).$$

**Proof.** Begin by recalling that $\bar{r}_{i,j}(\alpha) \geq \frac{3\alpha}{8}$ for all $(i, j) \in \mathbb{Z}^2$. This implies that

$$\frac{\bar{r}_{i',j'}(\alpha)}{\bar{r}_{i,j}(\alpha)} \leq 3,$$

and hence

$$\|G^{(2)}(\psi, \alpha)\|_1 \leq 3 \sum_{(i,j) \in \mathbb{Z}^2} \sum_{i' \neq i, j' \neq j} |\sin(\bar{\theta}_{i',j'}(\alpha) + \psi_{i',j'} - \bar{\theta}_{i,j}(\alpha) - \psi_{i,j}) - \sin(\bar{\theta}_{i',j'}(\alpha) - \bar{\theta}_{i,j}(\alpha))|$$

$$- \cos(\bar{\theta}_{i',j'}(\alpha) - \bar{\theta}_{i,j}(\alpha))(\psi_{i',j'} - \psi_{i,j})|. $$

The bound stated in the lemma now follows through a simple application of Taylor’s theorem as in [3, Lemma 5.1].

**Lemma 5.8.** There exists a constant $C_{G,3} > 0$ such that for all $\alpha \in [0, \alpha^*]$ and $s \in \ell^1$ with $\|s\|_{\infty} \leq \frac{3a}{8}$ we have

$$\|G^{(3)}(s, \alpha)\|_1 \leq C_{G,3}(\|s\|_{q^*} + \|s\|_2^3).$$

**Proof.** As in the proof of Lemma 5.6, we remark that the restriction $\|s\|_{\infty} \leq \frac{3a}{8}$ guarantees that the terms $\bar{r}_{i,j}(\alpha) + s_{i,j}$ are uniformly bounded in absolute value away from zero. Since $\omega_1$ is twice continuously differentiable we find that

$$|\omega_1(\bar{r}_{i,j}(\alpha) + s_{i,j}, \alpha) - \omega_1(\bar{r}_{i,j}(\alpha), \alpha)| \leq M_\omega(r_{i,j} + s_{i,j})|s_{i,j}|,$$

for all $(i, j) \in \mathbb{Z}^2$, where we have defined

$$M_\omega(r_{i,j} + s_{i,j}) := \max\{|\partial_1\omega_1(\bar{r}_{i,j} + R, \alpha)| : |R| \leq s_{i,j}, \alpha \in [0, \alpha^*]\}.$$ We note that $M_\omega : \mathbb{R} \to \mathbb{R}$ is well-defined and continuous. Then, recalling that Hypothesis 1 gives $\partial_1\omega_1(a, \alpha) = 0$ for all $\alpha \geq 0$, we find that there exists a uniform constant $C_\omega > 0$ such that

$$|\partial_1\omega_1(r_{i,j} + R, \alpha)| = |\partial_1\omega_1(r_{i,j} + R, \alpha) - \partial_1\omega_1(a, \alpha)| \leq C_\omega|r_{i,j} - a + R|,$$

for all $|R| \leq |s_{i,j}| \leq \frac{3a}{8}$ and $\alpha \in [0, \alpha^*]$. Hence,

$$M_\omega(r_{i,j} + s_{i,j}) \leq C_\omega \cdot \max\{|r_{i,j} - a + R| : |R| \leq s_{i,j}\} \leq C_\omega(\|r_{i,j} - a\| + |s_{i,j}|).$$

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Therefore, these estimates give
\[ \|G(3)(s, \alpha)\|_1 \leq \sum_{(i,j) \in \mathbb{Z}^2} |\omega_1(\bar{r}_{i,j}(\alpha) + s_{i,j}, \alpha) - \omega_1(\bar{r}_{i,j}(\alpha), \alpha)| \]
\[ \leq \sum_{(i,j) \in \mathbb{Z}^2} M_\omega(\bar{r}_{i,j}(\alpha) + s_{i,j})|s_{i,j}| \]
\[ \leq C_\omega \sum_{(i,j) \in \mathbb{Z}^2} (|\bar{r}_{i,j}(\alpha) - a| + |s_{i,j}|)|s_{i,j}| \]
\[ \leq C_\omega \sum_{(i,j) \in \mathbb{Z}^2} |\bar{r}_{i,j}(\alpha) - a||s_{i,j}| + C_\omega \sum_{(i,j) \in \mathbb{Z}^2} |s_{i,j}|^2 \]
\[ \leq C_\omega \sum_{(i,j) \in \mathbb{Z}^2} |\bar{r}_{i,j}(\alpha) - a||s_{i,j}| + C_\omega \|s\|_2^2. \]

(5.12)

Then, using Hölder’s inequality and Hypothesis 4 we find that
\[ \sum_{(i,j) \in \mathbb{Z}^2} |r_{i,j}(\alpha) - a||s_{i,j}| \leq \left( \sum_{(i,j) \in \mathbb{Z}^2} |r_{i,j}(\alpha) - a|^p \right)^{\frac{1}{p}} \left( \sum_{(i,j) \in \mathbb{Z}^2} |s_{i,j}|^q \right)^{\frac{1}{q}} \leq C_{\text{sol}} \|s\|_q, \]
\[ \text{giving the desired estimates. The proof of the lemma now follows from putting (5.12) and (5.13) together, completing the proof of the lemma.} \]

We state the following result as a corollary of the previous three lemmas for the convenience of citing these results in the following section.

**Corollary 5.9.** There exists a constant $C_G > 0$ such that for all $\psi \in \ell^1$, $\alpha \in [0, \alpha^*]$, and $s \in \ell^1$ with $\|s\|_\infty \leq \frac{3a}{8}$ we have
\[ \|G(s, \psi, \alpha) - \tilde{L}_\alpha \psi\|_1 \leq C_G(\|s\|_q + \|s\|_2^2 + \|s\|_2 Q_2(\psi) + Q_2^2(\psi)). \]

### 5.4 Nonlinear Stability of the Phase Components

In this subsection we use the linear and nonlinear estimates of the previous two subsections to prove Theorem 3.2. We write (5.1) as
\[ \frac{d}{d \tau} \psi = \tilde{L}_\alpha \psi + [G(\sigma(\psi(\tau), \alpha) + \rho(\alpha^{-1} \tau), \psi(\tau), \alpha) - \tilde{L}_\alpha \psi(\tau)], \]
(5.14)
along with the initial condition $\psi(0) = \psi^0 \in \ell^1$. Then, using the variation of constants formula, (5.14) can be written equivalently as
\[ \psi(t) = e^{\tilde{L}_\alpha t} \psi^0 + \int_0^t e^{\tilde{L}_\alpha (t-u)} [G(\sigma(\psi(u), \alpha) + \rho(\alpha^{-1} u), \psi(u), \alpha) - \tilde{L}_\alpha \psi(u)] du. \]
(5.15)
It will be the equation (5.15) which will remain the focus throughout this subsection. We note that we can no longer take $\alpha = 0$ since the temporal variable $\alpha^{-1} \tau$ will be undefined, and therefore we only consider $\alpha > 0$ and small.
Let us now define the mapping, denoted $T_3$, by

$$T_3\psi(t) = e^{\tilde{L}_\alpha t}\psi^0 + \int_0^\tau e^{\tilde{L}_\alpha (t-u)}[G(\sigma(\psi(u),\alpha) + \rho(\alpha^{-1}u),\psi(u),\alpha) - \tilde{L}_\alpha \psi(u)]du,$$  \hspace{1cm} (5.16)

so that fixed points of $T_3$ are exactly the solutions of \ref{5.14} via the equivalent formulation \ref{5.15}. Then, prior to stating our results, we note that for all $\psi \in \ell^p$ and $p \in [1, \infty]$ we have

$$\|\sigma(\psi,\alpha) + \rho(t)\|_p \leq \|\sigma(\psi,\alpha)\|_p + \|\rho(t)\|_p \leq \sqrt{\alpha}Q_p(\psi) + \|\rho(t)\|_1,$$  \hspace{1cm} (5.17)

following from the Lipschitz properties of the invariant manifold and the monotonicity of $\ell^p$ norms. This leads to the following lemma.

**Lemma 5.10.** Let $\tau_0 > 0$ be arbitrary and assume that

$$Q_2(\psi(\tau)) \leq 2\varepsilon\tilde{C}_L(1 + \tau)^{-\min\left\{\frac{1}{2} + \frac{n}{p_0} + \eta\right\}}$$  \hspace{1cm} (5.18)

and

$$Q_{q^*}(\psi(\tau)) \leq 2\varepsilon\tilde{C}_L(1 + \tau)^{-\frac{p_0}{\beta} - \eta}$$  \hspace{1cm} (5.19)

for all $0 \leq \tau \leq \tau_0$, for some $\varepsilon > 0$. Then, there exists $\alpha_1, \varepsilon^* > 0$, independent of $\tau_0$, such that for all $\alpha \in (0, \alpha_1]$ and $\varepsilon \in (0, \varepsilon^*]$, if $\|\rho^0\|_1, \|\rho^0\|_1 \leq \varepsilon$ we have

$$Q_2(T_3\psi(\tau)) \leq 2\varepsilon\tilde{C}_L(1 + \tau)^{-\min\left\{\frac{1}{2} + \frac{n}{p_0} + \eta\right\}},$$

$$Q_{q^*}(T_3\psi(\tau)) \leq 2\varepsilon\tilde{C}_L(1 + \tau)^{-\frac{p_0}{\beta} - \eta},$$

for all $0 \leq \tau \leq \tau_0$.

**Proof.** We begin by noting that by restricting $\varepsilon \leq \delta^*$ we guarantee that $\|\rho^0\|_1 \leq \varepsilon$ implies that $\|\rho(\alpha^{-1}\tau)\|_1 \leq 2\varepsilon e^{-\frac{\alpha}{\delta^*}},$ from Theorem 3.1. Hence, we ensure that $\varepsilon^* \leq \delta^*$.

Then, we begin with the bound on $Q_2$. Using \ref{5.16} we obtain

$$Q_2(T_3\psi(\tau)) \leq Q_2(e^{\tilde{L}_\alpha \tau}\psi^0) + \int_0^\tau Q_2(e^{\tilde{L}_\alpha (\tau-u)}[G(\sigma(\psi(u),\alpha) + \rho(\alpha^{-1}u),\psi(u),\alpha) - \tilde{L}_\alpha \psi(u))]du,$$

for all $0 \leq \tau \leq \tau_0$, and from Proposition 5.3 and Corollary 5.9 we obtain

$$Q_2(T_3\psi(\tau)) \leq \tilde{C}_L(1 + \tau)^{\min\left\{\frac{1}{2} + \frac{n}{p_0} + \eta\right\}}\psi^0_1 + \tilde{C}_L CG \int_0^\tau (1 + \tau - u)^{-\min\left\{\frac{1}{2} + \frac{n}{p_0} + \eta\right\}}\|\sigma(\psi(u),\alpha) + \rho(\alpha^{-1}u)\|_q^\prime du$$

$$+ \tilde{C}_L CG \int_0^\tau (1 + \tau - u)^{-\min\left\{\frac{1}{2} + \frac{n}{p_0} + \eta\right\}}\|\sigma(\psi(u),\alpha) + \rho(\alpha^{-1}u)\|_2^2 du$$

$$+ \tilde{C}_L CG \int_0^\tau (1 + \tau - u)^{-\min\left\{\frac{1}{2} + \frac{n}{p_0} + \eta\right\}}\|\sigma(\psi(u),\alpha) + \rho(\alpha^{-1}u)\|_2 Q_2(\psi(u)) du$$

$$+ \tilde{C}_L CG \int_0^\tau (1 + \tau - u)^{-\min\left\{\frac{1}{2} + \frac{n}{p_0} + \eta\right\}}Q_2^2(\psi(u)) du,$$  \hspace{1cm} (5.20)
where $\tilde{C}_L > 0$ is the constant coming from Proposition 5.3 and $C_G > 0$ is the constant coming from Corollary 5.9. We now bound each integral in (5.20) separately, and bring them together at the end.

First, using (5.17) we get

\[
\int_0^\tau (1 + \tau - u)^{-\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}} \|\sigma(\psi(u), \alpha) + \rho(\alpha^{-1} u)\|_q \cdot du \leq \sqrt{\alpha} \int_0^\tau (1 + \tau - u)^{-\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}} Q_{\eta^*}(\psi(u)) du + \int_0^\tau (1 + \tau - u)^{-\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}} \|\rho(\alpha^{-1} u)\|_1 du
\]

Then, recalling that $\|\rho(\alpha^{-1} \tau)\|_1 \leq 2\varepsilon e^{-\frac{2}{p^*} \tau}$ for all $0 \leq \tau \leq \tau_0$, we may apply Lemma 5.2 to find that there exists a constant $C_{\frac{1}{2} + \eta, \beta} > 0$ such that

\[
\int_0^\tau (1 + \tau - u)^{\frac{1}{2} - \eta^*} Q_{\eta^*}(\psi(u)) du \leq 2\alpha\varepsilon C_{\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}, \beta}(1 + \tau)^{-\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}},
\]

for all $0 \leq \tau \leq \tau_0$. Then, the bound (5.19) then allows one to apply Lemma 5.1 to find that

\[
\int_0^\tau (1 + \tau - u)^{-\frac{1}{2} - \eta^*}(1 + u)^{-\frac{1}{p^*} - \eta} du \leq 2\sqrt{\alpha\varepsilon C^{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta}}(1 + \tau)^{-\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}}.
\]

Since $\frac{1}{p^*} + \eta > 1$ from Hypothesis 4. Putting this together gives the bound

\[
\int_0^\tau (1 + \tau - u)^{-\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}} \|\sigma(\psi(u), \alpha) + \rho(\alpha^{-1} u)\|_q, du \leq 2\sqrt{\alpha\varepsilon}[2\sqrt{\alpha\delta^* C_{\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}, \beta} + \tilde{C}_LC_{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta}}](1 + \tau)^{-\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}}.
\]

Now, turning to the second integral in (5.20), we first note that

\[
\|\sigma(\psi(\tau), \alpha) + \rho(\alpha^{-1} \tau)\|_2^2 \leq \alpha Q_2^2(\psi(\tau)) + \sqrt{\alpha} Q_2(\psi(\tau))\|\rho(\alpha^{-1} \tau)\|_1 + \|\rho(\alpha^{-1} \tau)\|_2^2 \leq \alpha Q_2(\psi(\tau)) + 4\alpha\varepsilon^2 \tilde{C}_L e^{-\frac{2}{p^*} \tau} + 4\alpha\varepsilon^2 e^{-\frac{2\alpha}{p^*} \tau}
\]

since $Q_2(\psi(\tau)) \leq 2\varepsilon \tilde{C}_L$ for all $0 \leq \tau \leq \tau_0$. Then, using Lemmas 5.1 and 5.2 the above bound implies that

\[
\int_0^\tau (1 + \tau - u)^{-\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}} \|\sigma(\psi(u), \alpha) + \rho(\alpha^{-1} u)\|_2^2 du \leq 4\alpha\varepsilon C^2 L \int_0^\tau (1 + \tau - u)^{-\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}} (1 + u)^{-2\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}} du + 4\alpha\varepsilon^2(\sqrt{\alpha\tilde{C}_L C_{\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}, \beta} + \tilde{C}_LC_{\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}, 2\beta})(1 + \tau)^{-\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}})
\]

\[
+ 4\alpha\varepsilon^2(\sqrt{\alpha\tilde{C}_L C_{\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}, \beta} + \tilde{C}_LC_{\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}, 2\beta})(1 + \tau)^{-\min\{\frac{1}{2} + \eta, \frac{1}{p^*} + \eta\}})
\]

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for the same constant $C_{\frac{1}{2}+\eta,\beta} > 0$ used previously, and another pair of positive constants $C_{\min\{\frac{1}{2}+\eta,\frac{1}{p^*}+\eta\},2\min\{\frac{1}{2}+\eta,\frac{1}{p^*}+\eta\}}$ and $C_{\frac{1}{2}+\eta,2\beta}$ since $2\min\{\frac{1}{2}+\eta,\frac{1}{p^*}+\eta\} > 1$.

Very similar manipulations to those used to bound (5.21) and (5.22) yield the bounds

$$\int_0^\tau (1 + \tau - u)^{-\min\{\frac{1}{2}+\eta,\frac{1}{p^*}+\eta\}}||\sigma(u),\alpha||_2Q_2(\psi(u))du \leq 2\sqrt{\epsilon C_L(2C_LC_{\min\{\frac{1}{2}+\eta,\frac{1}{p^*}+\eta\},2\min\{\frac{1}{2}+\eta,\frac{1}{p^*}+\eta\})}Q_2(\psi(u))(1 + \tau)^{-\min\{\frac{1}{2}+\eta,\frac{1}{p^*}+\eta\}}, \quad (5.23)$$

and

$$\int_0^\tau (1 + \tau - u)^{-\min\{\frac{1}{2}+\eta,\frac{1}{p^*}+\eta\}}Q_2^2(\psi(u))du \leq 4\epsilon C_L(2C_L^2C_{\min\{\frac{1}{2}+\eta,\frac{1}{p^*}+\eta\},2\min\{\frac{1}{2}+\eta,\frac{1}{p^*}+\eta\})Q_2(\psi(u))(1 + \tau)^{-\min\{\frac{1}{2}+\eta,\frac{1}{p^*}+\eta\}}, \quad (5.24)$$

Putting (5.21) and (5.24) into (5.20) gives the bound

$$Q_2(T_3\psi(\tau)) \leq (\epsilon(\psi^0)||1 + \epsilon h(\alpha,\epsilon))(1 + \tau)^{-\min\{\frac{1}{2}+\eta,\frac{1}{p^*}+\eta\}}$$

where we have introduced the function $h(\alpha,\epsilon)$ to collect all the constants coming from (5.20) and (5.21). The most important point is that $h(0,0) = 0$ and that $h$ depends continuously on $\alpha,\epsilon \geq 0$. Hence, for sufficiently small $\alpha,\epsilon \geq 0$ one can guarantee that $h(\alpha,\epsilon) \leq C_L$, giving the bound on $Q_2(U\psi(\tau))$. The bound on $Q_9(U\psi(\tau))$ follows via a nearly identical manipulation and is therefore omitted.

**Lemma 5.11.** For all $\alpha \in (0,\alpha_1]$ and $\epsilon \in [0,\epsilon^*]$ and $||\psi^0||_1,||\rho^0||_1 \leq \epsilon$ there exists a unique fixed point of $T_3$, denoted $\psi(\tau)$, such that $\psi(0) = \psi^0$ and

$$Q_2(\psi(\tau)) \leq 2\epsilon C_L(1 + \tau)^{-\min\{\frac{1}{2}+\eta,\frac{1}{p^*}+\eta\}}, \quad (5.25)$$

for all $\tau \geq 0$.

**Proof.** This proof proceeds in an identical manner to [3] Theorem 4.5, and therefore we will merely describe the important points which must be considered. The proof is simply a bootstrapping argument via an application of the Banach fixed point theorem. One begins by obtaining a fixed point of $T_3$ for all $\tau \in [0,1]$ by using Lemma 5.10 to show that $T_3$ is a well-defined mapping on a complete metric space of functions satisfying (5.18) and (5.19) with $\tau_0 = 1$. Upon obtaining such a fixed point with the Banach fixed point theorem, we extend this solution to $\tau \in [0,2]$ by defining a new complete metric space of functions satisfying (5.18) and (5.19) for $\tau_0 = 2$, such that these functions agree with the fixed point of $T_3$ on $[0,1]$. Again, Lemma 5.10 gives that $T_3$ would be a well-defined mapping in this case. The argument then proceeds inductively to show that if one has a fixed point on $[0,n]$ for any integer $n \geq 1$, then it can be extended to a fixed point on $[0,n+1]$ satisfying the decay rates (5.18) and (5.19). This argument therefore gives the proof of the lemma. \(\square\)
Having now obtained a fixed point of $T_3$ satisfying the decay rates \((5.25)\), we are now able to prove that the other decay rates of Theorem 3.2 follow. The results of Lemma 5.11 and Corollary 5.12 therefore finish the proof of Theorem 3.2. The important point to note is that the following corollary dictates that the other decay rates of Theorem 3.2 follow. The results of Lemma 5.11 and Corollary 5.12 therefore the nonlinear terms are only bounded by the rates stated in Theorem 3.2. Of course, this should be apparent to the reader since Corollary 5.9 states the nonlinear terms are only bounded by $Q_2(\psi(\tau))$ and $Q_\sigma(\psi(\tau))$.

**Corollary 5.12.** There exists a constant $C_\psi > 0$ such that for all $\alpha \in (0, \alpha_1]$ and $\varepsilon \in [0, \varepsilon^*]$ and $\|\psi^0\|_1, \|\rho^0\|_1 \leq \varepsilon$ the unique fixed point of $T_3$ satisfying $\psi(0) = \psi^0$ and \((5.25)\) further satisfies

$$
\|\psi(t)\|_p \leq \varepsilon C_\psi (1 + \alpha t)^{−1+\frac{1}{p}},
$$

$$
Q_p(\psi(t)) \leq \varepsilon C_\psi (1 + \alpha t)^{−\min\{1−\frac{1}{p} + \eta, \frac{2}{p} + \eta\}},
$$

for all $\tau \geq 0$, $\alpha \in [0, \alpha_1]$, and $p \in [1, \infty]$.

**Proof.** First, since we assume $\psi(\tau)$ is a fixed point of $T_3$, then it necessarily satisfies \((5.15)\). Then, for $p \in [1, \infty]$ fixed, from Proposition 5.3 and Corollary 5.9

$$
\|\psi(\tau)\|_p \leq (1 + \alpha t)^{−1+\frac{1}{p}}\|\psi^0\|_1 + \tilde{C}_L C_G \int_0^\tau (1 + \tau - u)^{−1+\frac{1}{p}} \|\sigma(u, \alpha) + \rho(\alpha^{-1}u)\|_q du
$$

$$
+ \tilde{C}_L C_G \int_0^\tau (1 + \tau - u)^{−1+\frac{1}{p}} \|\sigma(\psi(u), \alpha) + \rho(\alpha^{-1}u)\|_2 du
$$

$$
+ \tilde{C}_L C_G \int_0^\tau (1 + \tau - u)^{−1+\frac{1}{p}} \|\sigma(\psi(u), \alpha) + \rho(\alpha^{-1}u)\|_2 Q_2(\psi(u)) du
$$

$$
+ \tilde{C}_L C_G \int_0^\tau (1 + \tau - u)^{−1+\frac{1}{p}} Q_\sigma^2(\psi(u)) du.
$$

From here bounding each of the integrals is nearly identical to the bounds \((5.21), (5.24)\), with the notable exceptions being that the rate of decay will now be $−1 + \frac{1}{p}$ and the constants obtained from Lemmas 5.1 and 5.2 will reflect these different decay rates. By the continuity of the constants from Lemmas 5.1 and 5.2 they are uniformly bounded in $p \in [1, \infty]$ and hence one obtains the desired bound on $\|\psi(t)\|_p$. A similar manipulation yields the bounds on $Q_p(\psi(t))$. \(\square\)

### 6 Determining $\eta > 0$

Here we provide a heuristic argument that $\eta = 1$ in Hypothesis 3 although we provide no formal proof and therefore this remains an open topic of inquiry left for future investigations. The existence of a value $\eta > 0$ can be confirmed via a series of sufficient conditions outlined in [3], but this work gives no indication of what exactly its value is or how it relates to the dimension of the underlying lattice. The reason for this is due to the fact that the existence of $\eta$ comes from a more theoretical investigation into random walks on infinite weighted graphs which gives no indication as to what the particular value of $\eta$ should be in our present situation [19]. Hence, in the absence of a formal proof, we are left to speculate as to what exactly $\eta$ should be. This will be the focus of this section.
Much of the study of lattice dynamical systems has been informed by the behaviour of solutions to their spatially continuous partial differential equation counterparts. Therefore, we begin with the heat equation posed on the infinite two-dimension plane. That is, we consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

(6.1)

where \( u = u(x, y, t) : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R} \). We further consider the continuous spatial analogues of the \( \ell^p \) spaces given in (2.1), written \( L^p \), and defined as

$$L^p = \left\{ f : \mathbb{R}^2 \to \mathbb{R} : \int_{\mathbb{R}^2} |f(x, y)|^p \, dx \, dy < \infty \right\}$$

for all \( p \in [1, \infty) \), along with the space \( L^\infty \) given by

$$L^\infty = \left\{ f : \mathbb{R}^2 \to \mathbb{R} : \sup_{(x, y) \in \mathbb{R}^2} |f(x, y)| < \infty \right\}.$$

The norm

$$\|f\|_{L^p} := \left( \int_{\mathbb{R}^2} |f(x, y)|^p \, dx \, dy \right)^{\frac{1}{p}}$$

makes \( (L^p, \| \cdot \|_{L^p}) \) a Banach space for all \( p \in [1, \infty) \), and similarly, the norm

$$\|f\|_{L^\infty} := \sup_{(x, y) \in \mathbb{R}^2} |f(x, y)|$$

makes \( (L^\infty, \| \cdot \|_{L^\infty}) \) a Banach space.

Now, given a function \( f : \mathbb{R}^2 \to \mathbb{R} \), the solution \( u(x, y, t) \) to (6.1) with the initial condition \( u(x, y, 0) = f(x, y) \) is given by

$$u(x, y, t) = \frac{1}{4\pi t} \int_{\mathbb{R}^2} e^{-\frac{(x-x')^2+(y-y')^2}{4t}} f(x', y') \, dx' \, dy'. $$

If \( f \in L^1 \) we may apply Hölder’s inequality to find that

$$\|u(x, y, t)\|_{L^\infty} \leq \frac{1}{4\pi t} \|f\|_{L^1},$$

since

$$e^{-\frac{(x-x')^2+(y-y')^2}{4t}} \leq 1$$

for all \( (x, y), (x', y') \in \mathbb{R}^2 \) and \( t \geq 0 \). Taking the partial derivative of \( u(x, y, t) \) with respect to \( x \), denoted \( u_x(x, y, t) \), gives

$$u_x(x, y, t) = \frac{1}{8\pi t^2} \int_{\mathbb{R}^2} (x - x') e^{-\frac{(x-x')^2+(y-y')^2}{4t}} f(x', y') \, dx' \, dy', $$

which again after applying Hölder’s inequality we can find that

$$\|u_x(x, y, t)\|_{L^\infty} \leq \frac{1}{8\sqrt{2\pi t^2}} \|f\|_{L^1},$$
since
\[ |x - x'|e^{-\frac{(x-x')^2 + (y-y')^2}{4t}} \leq \frac{1}{\sqrt{2e}}, \]
for all \((x, y), (x', y') \in \mathbb{R}^2\) and \(t \geq 0\). We may obtain the same bound for \(\|u_y(x, y, t)\|_{L^\infty}\), and hence we find that
\[ \|u_x(x, y, t)\| + |u_y(x, y, t)||_{L^\infty} \leq \frac{1}{4\sqrt{2e}t^2} \|f\|_{L^1}, \]
which is the continuous spatial analogue of \(Q_\infty\). We note that this gives a gradient decay at a rate of \(t^{-2}\), and therefore in the continuous spatial setting we find that \(\eta = 1\).

We now turn to the spatially discrete counterpart to (6.1) given by
\[ \dot{u}_{i,j} = (u_{i+1,j} - u_{i,j}) + (u_{i-1,j} - u_{i,j}) + (u_{i,j+1} - u_{i,j}) + (u_{i,j-1} - u_{i,j}) = \sum_{i',j'}(u_{i',j'} - u_{i,j}), \quad (i, j) \in \mathbb{Z}^2, \quad (6.2) \]
where \(u_{i,j} = u_{i,j}(t), \ t \geq 0\), and we have returned to the notation \(\dot{u}_{i,j}\) to represent differentiation with respect to \(t\) since \(u_{i,j}\) is now simply a function of a single variable. Then, given an initial condition \(u^0 \in \ell^1\) it is known that the solution to (6.2), denoted \(u(t)\), with \(u(0) = u^0\) behaves similarly to its continuous spatial counterpart in that
\[ \|u(t)\|_{L^\infty} \leq Ct^{-1}\|u^0\|_1, \]
for some constant \(C > 0\). Moreover, it is through the behaviour of solutions to (6.2) that one is able to infer the decay rates (3.15) for linear lattice equations of the form
\[ \dot{u} = L_\alpha u. \quad (6.3) \]
This is obtained by demonstrating that the underlying graphs related to the right-hand sides of (6.2) and (6.3) are roughly isomorphic to each other [3, Section 6]. The most important point is that much, if not all, of the behaviour of solutions to (6.3) can be inferred by the behaviour of solutions to (6.2) when a rough isometry is established [21].

Since we see that solutions to both the continuous and discrete spatial heat equations decay pointwise at a uniform rate of \(t^{-1}\), one would expect that the discrete gradient of solutions to the discrete heat equation (6.2) behave similarly as well. That is, we expect that if \(u(t)\) is a solution to (6.2) with \(u(0) = u^0\), a logical hypothesis would be that there exists \(\tilde{C} > 0\) such that
\[ Q_\infty(u(t)) \leq \tilde{C} t^{-2}\|u^0\|_1. \]
Such a decay rate would then give that \(\eta = 1\). Furthermore, having established a rough isometry, one expects that these decay properties should be transferred over to solutions of (6.3) with initial conditions \(u^0 \in \ell^1\) as well.

Of course, none of the discussion in this section should substitute a formal proof, and therefore we can only conjecture that \(\eta = 1\) in our present situation. This therefore necessitates further investigations into what exactly this value of \(\eta\) is, and how it relates to the dimension of the underlying lattice. Moreover, the value of \(\eta\) plays a crucial role in Hypothesis [4] since we require a value \(p^* \in [1, \infty)\) such that
\[ \frac{1}{p^*} + \eta > 1. \]
Should it be the case that \(\eta = 1\), we see that any finite \(p^* \geq 1\) will suffice, greatly simplifying this assumption.
7 Application to Rotating Waves

In this section we focus on the particular case of system (3.1) given by

\[ \begin{aligned}
\dot{z}_{i,j} &= \alpha \sum_{i',j'} (z_{i',j'} - z_{i,j}) + z_{i,j} (1 + \omega_0(\alpha) + |z_{i,j}|^2), \quad (i,j) \in \mathbb{Z}^2, \\
\end{aligned} \]

(7.1)

so that \( \omega_1 \equiv 0 \). Introducing the polar ansatz (3.6) we arrive at the related polar systems of equations

\[ \begin{aligned}
\dot{r}_{i,j} &= \alpha \sum_{i',j'} (r_{i',j'} \cos(\theta_{i',j'} - \theta_{i,j}) - r_{i,j}) + r_{i,j} (1 - r_{i,j}^2), \\
\dot{\theta}_{i,j} &= \alpha \sum_{i',j'} \frac{r_{i',j'}}{r_{i,j}} \sin(\theta_{i',j'} - \theta_{i,j}), \\
\end{aligned} \]

(7.2)

for all \((i,j) \in \mathbb{Z}^2\). Clearly our system (3.1) satisfies Hypothesis 1 with \( \alpha = 1 \).

As previously remarked, it was shown in [2] that system (3.1) possesses a particular nontrivial solutions which resembles a rotating wave from the continuous spatial context which satisfies Hypothesis 2 for a sufficiently small \( \alpha > 0 \). Throughout this section we simply denote this rotating wave solution by \( \{(\bar{r}_{i,j}(\alpha), \bar{\theta}_{i,j}(\alpha))\}_{(i,j) \in \mathbb{Z}^2} \) for convenience. In the present discrete spatial context a rotating wave solution is identified by the discrete rotational identity

\[ \begin{aligned}
z_{j,1-i}(t; \alpha) &= e^{i\frac{\pi}{2}} \cdot z_{i,j}(t; \alpha), \\
\end{aligned} \]

(7.3)

for every \((i,j) \in \mathbb{Z}^2\) and \( \alpha \in [0, \alpha^*] \), upon returning back to the single complex variable \( z_{i,j} \) via the ansatz (3.6). The meaning of the identity (7.3) is that a rotation of the entire lattice clockwise through an angle of \( \pi/2 \) about a theoretical centre cell at \( i = j = \frac{1}{2} \) simply leads to a phase advance of a quarter period. Indeed, the ansatz (3.6) gives

\[ \begin{aligned}
e^{i\frac{\pi}{2}} \cdot z_{i,j}(t; \alpha) &= \bar{r}_{i,j}(\alpha) e^{i(\omega_0(\alpha)t + \bar{\theta}_{i,j}(\alpha) + \frac{\pi}{2})} \\
&= \bar{r}_{i,j}(\alpha) e^{i(\omega_0(\alpha)(t + \frac{\pi}{2\omega_0(\alpha)}) + \bar{\theta}_{i,j}(\alpha))} \\
&= z_{i,j}(t + T(\alpha)/4; \alpha), \\
\end{aligned} \]

where \( T(\alpha) := 2\pi/\omega_0(\alpha) \) is the period of the periodic solution. Here the theoretical centre cell at \( i = j = \frac{1}{2} \) acts as the centre of rotating for the rotating wave. An example of the solution \( \{(\bar{r}_{i,j}(\alpha), \bar{\theta}_{i,j}(\alpha))\}_{(i,j) \in \mathbb{Z}^2} \) obtained on a 100 \( \times \) 100 lattice with Neumann boundary conditions is shown in Figure 1. One finds that the values of the phases, \( \{\bar{\theta}_{i,j}(\alpha)\}_{(i,j) \in \mathbb{Z}^2} \), around each concentric ring about the centre four cell ring at the indices \((i,j) = (0,0), (0,1), (1,0), (1,1)\) increase from \(-\pi\) up to \( \pi \) monotonically. This was proven for \( \alpha = 0 \) in [1], and numerical investigations on the finite lattice lead one to conjecture that this holds for all \( \alpha > 0 \) for which the solution exists. Most importantly, we have indeed confirmed that Hypotheses [1] and [2] do indeed hold for system (7.1) and this rotating wave solution, giving that the results of Theorem 3.1 hold for this rotating wave solution.

We now turn to verifying Hypothesis [3] The decay rates (3.15) were shown to be true for \( \alpha = 0 \) in [3] Section 6.2], and these arguments can be replicated to show that the decay rates (3.15) hold for sufficiently small \( \alpha \geq 0 \). To verify this claim, we begin by noting that the work in [2] gives that the
The radial and phase components of a rotating wave solution to \( \dot{\phi} = 1 \) on a finite \( 100 \times 100 \) lattice with Neumann boundary conditions. The four cell ring given by the indices \((i, j) = (0, 0), (0, 1), (1, 0), (1, 1)\) acts as the centre of the rotating wave, and on the left one sees that the radial components show very little deviation from the unique positive root at \( R = 1 \) of \( \lambda(R) = (1 - R^2) \) as one moves out from this centre of the rotating wave. Around each concentric ring about the centre four cell ring, the phase components increase from \(-\pi\) up to \(\pi\) monotonically, which can be observed in the contour plot of the phase components on the right.

The coupling between any two of the four centre cells is exactly \( \pi/2 \) since we have

\[
\bar{\theta}_{1,1}(\alpha) = 0,
\bar{\theta}_{0,1}(\alpha) = \frac{\pi}{2},
\bar{\theta}_{0,0}(\alpha) = \pi,
\bar{\theta}_{1,0}(\alpha) = \frac{3\pi}{2},
\]

for all \( \alpha > 0 \) for which the solution exists. At \( \alpha = 0 \) we have that all other nearest-neighbour interactions are such that \(|\bar{\theta}_{i,j}(0) - \bar{\theta}_{i,j}(0)| < \frac{\pi}{2}\), thus giving that continuity with respect to \( \alpha \) will ensure that for sufficiently small \( \alpha > 0 \) we have

\[
\cos(\bar{\theta}_{i',j'}(\alpha) - \bar{\theta}_{i,j}(\alpha)) > 0,
\]

for all \((i, j), (i', j') \notin \{(0, 0), (0, 1), (1, 0), (1, 1)\}\). We then consider a graph with vertices given by the indices of the lattice \( \mathbb{Z}^2 \) together with an edge set which connects nearest-neighbours if, and only if, the quantity \( \cos(\bar{\theta}_{i,j}(\alpha) - \bar{\theta}_{i,j}(\alpha)) \) is positive, and no other edges present in the graph. Visually, this graph is simply the standard integer lattice with edges connecting all nearest-neighbours less those between the four centre cells, as is shown in Figure 2. Our arguments above imply that for sufficiently small \( \alpha \geq 0 \) the geometry of this graph remains unchanged in that no new edges are created or destroyed. This understanding of the graph geometry for sufficiently small \( \alpha \geq 0 \) then allows one to follow the arguments of [3, Section 6.2] to confirm Hypothesis 3 for this rotating wave solution.

It therefore remains to confirm that Hypothesis 4 holds for the rotating wave solution. Unfortunately a complete analytic proof remains elusive, and therefore we restrict ourselves to numerical investigations.
in an effort to at least conjecture that this hypothesis does indeed hold for this rotating wave solution. To begin, define the following quantities

\[
M_r^p(\alpha) = \sum_{(i,j) \in \mathbb{Z}^2} |r_{i,j}(\alpha) - 1|^p,
\]
\[
M_\theta^p(\alpha) = \sum_{(i,j) \in \mathbb{Z}^2} \sum_{i',j'} |\sin(\bar{\theta}_{i',j'}(\alpha) - \bar{\theta}_{i,j}(\alpha))|^p.
\] (7.4)

Using MATLAB we are able to generate the rotating wave solution on a finite \(N \times N\) lattice with Neumann boundary conditions, for increasing \(N\), in an effort to conjecture that there exists \(p \geq 1\) such that \(M_r^p(\alpha), M_\theta^p(\alpha) < \infty\) for sufficiently small \(\alpha \geq 0\).

In Figure 3 we provide plots of the quantities \(M_r^1(\alpha)\) and \(M_\theta^1(\alpha)\) for \(\alpha = 0.1, 0.5, 1.0\) for the rotating wave solution simulated on a lattice of size \(N \times N\), with \(N\) increasing by ten from \(N = 10\) to \(N = 200\). From Figure 3 one infers that the quantities \(M_r^1(\alpha)\) and \(M_\theta^1(\alpha)\) should not be finite in the infinite lattice limit \((N \to \infty)\), and hence we should not expect \(p^* = 1\) in Hypothesis 4. This further presents a slight problem in our analysis since when \(p^*> 1\), we are required to know the exact value of \(\eta > 0\) to ensure that \(\frac{1}{p^*} + \eta > 0\) to apply Theorem 3.2. We remark that the work in the previous section leads one to expect that \(\eta = 1\), and therefore any finite \(p^* \geq 1\) will give \(\frac{1}{p^*} + \eta > 0\), allowing for the application of Theorem 3.2 to the rotating wave solution under investigation here.

Since we have conjectured that \(M_r^1(\alpha)\) and \(M_\theta^1(\alpha)\) are unbounded, Figure 4 provides analogous plots to Figure 3 for \(M_r^5(\alpha)\) and \(M_\theta^5(\alpha)\). The choice for providing approximations of \(M_r^p(\alpha)\) and \(M_\theta^p(\alpha)\) with \(p = 5\) is simply due to the fact that we have chosen \(p > 1\) sufficiently large to see that the plots appear to level off at relatively small lattice sizes. Figure 4 leads one to conjecture that upon moving to the infinite lattice limit both \(M_r^1(\alpha)\) and \(M_\theta^1(\alpha)\) are finite, at least for the simulated values \(\alpha = 0.1, 0.5, 1.0\). We remark that the approximations of \(M_r^5(0.1)\) appears to be slightly increasing with \(N\), but the numerics lead one to believe that this slight increase is attributed to numerical error in calculating the solution on larger and larger lattices. That is, we use the method outlined in \[32\] to
Figure 3: Provided are plots of the quantities $M_r^1(\alpha)$ and $M_\theta^1(\alpha)$, defined in (7.4), for $\alpha = 0.1, 0.5, 1.0$ for the rotating wave solution simulated on a lattice of size $N \times N$. From these plots one conjectures that as $N \to \infty$ these quantities become unbounded.

Figure 4: Provided are plots of the quantities $M_r^5(\alpha)$ and $M_\theta^5(\alpha)$, defined in (7.4), for $\alpha = 0.1, 0.5, 1.0$ for the rotating wave solution simulated on a lattice of size $N \times N$. From these plots one conjectures that as $N \to \infty$ these quantities remain bounded.
simulate the rotating wave solution which simply provides initial conditions that guarantee the phases converge to the solution. These initial conditions are quite far from the final state, thus taking a longer and longer amount of time to converge to equilibrium as the size of the lattice increases. Moreover, the rate of convergence decreases as the size of the lattice increases, which can be attributed to the shrinking of the spectral gap as $N \rightarrow \infty$. Although not as noticeable, this is also observed in $M^p_0(0.5)$ and $M^p_0(1)$ as well, but with a significantly smaller rate of increase from one lattice size to the next. Hence, we see that numerical investigations, although illuminating, are not enough to provide a definite answer as to whether Hypothesis \ref{h:main} is true for the rotating wave solution in question.

Since the numerics above could introduce a slight doubt into the mind of the reader, we conclude this section with a heuristic argument that $M^p_\theta(\alpha)$ should in fact be finite for sufficiently small $\alpha \geq 0$. In \cite{[11]} the author heuristically regards a rotating wave solution in this discrete spatial context as a collection of nested rings, so that a $2N \times 2N$ array is comprised of $N$ concentric rings. The inner core is comprised of four elements (representing the centre of the rotating wave described above), the next wraps around this inner core and has length twelve, and so on. This analogy easily extends to the case when $N \rightarrow \infty$, where we now have a countable infinity of nested rings concentrically wrapped around each other, starting with the inner core of the centre four cells. Then, each concentric ring has $l_n = 8n - 4$ elements for $n = 1, 2, 3, \ldots$. When these concentric rings are uncoupled from one another we can find a rotating wave solution such that the difference between two consecutive phases in the ring is exactly $\pm \frac{2\pi}{l_n}$.

In \cite{[11]} the author argues that when the concentric rings are coupled together, the solution should be similar. Indeed, in numerical simulations one can observe that as one moves away from the centre of rotation, the solution about concentric rings appear to converge to a solution for which the difference between two consecutive phases in the ring is exactly $\pm \frac{2\pi}{l_n}$. If this were true, then we would have

$$|\tilde{\theta}_{i',j'}(\alpha) - \tilde{\theta}_{i,j}(\alpha)| = \mathcal{O}\left(\frac{1}{n}\right),$$

for all four nearest-neighbours $(i', j')$, assuming that $\tilde{\theta}_{i,j}(\alpha)$ belongs to the ring of length $l_n$. This estimate would give that

$$\sum_{i',j'} |\sin(\tilde{\theta}_{i',j'}(\alpha) - \tilde{\theta}_{i,j}(\alpha))| = \mathcal{O}\left(\frac{1}{n^p}\right),$$

for all $(i, j) \in \mathbb{Z}^2$, again assuming that $\tilde{\theta}_{i,j}(\alpha)$ belongs to the ring of length $l_n$. Therefore, since $l_n = \mathcal{O}(n)$, summing over all $(i, j) \in \mathbb{Z}^2$ would give that $M^p_\theta(\alpha)$ is comparable to the series

$$\sum_{n=1}^\infty \frac{1}{n^{p-1}},$$

since we first sum over all $\tilde{\theta}_{i,j}(\alpha)$ belonging to each ring $l_n$, giving the $1/n^{p-1}$ term, and then over all rings indexed by $n \geq 1$. Hence, this would imply that to satisfy Hypothesis \ref{h:main} we require $p^* > 2$.

\section{Discussion}

In this work we have provided a series of sufficient conditions that demonstrate the local asymptotic stability of periodic solutions to our Lambda-Omega lattice dynamical system. Recall that our first
assumption, Hypotheses 1 simply states that our Lambda-Omega system should generalized a spatially
discretized Ginzburg-Landau equation and that Hypothesis 2 assumes the existence of a periodic solution
that exists for sufficiently small positive coupling values \( \alpha \geq 0 \). These two assumptions alone are all that
are required to demonstrate the existence of an invariant slow manifold which is locally asymptotically
stable with uniform exponential rate of decay. Although this result is certainly not surprising, it is
necessary for our understanding of local asymptotic stability in this Lambda-Omega setting, as well as
demonstrates a useful extension of Hale’s integral manifold theorems to the infinite-dimensional lattice
dynamical system context.

Upon proving the existence of a locally asymptotically stable invariant manifold to our Lambda-
Omega system, we turned our attention to investigating the behaviour of trajectories starting near
our periodic orbit on this manifold. In order to do so we required Hypothesis 3 which assumed the
decay of a certain semigroup with infinitesimal generator closely related to the linearization about the
periodic solution on the invariant manifold. We recall that these decay rates were not arbitrary, but
can be confirmed via the methods of [3] Section 6 through an understanding of the geometry of an
associated infinite weighted graph. Hypotheses 1-3 are relatively easy to confirm when compared to the
final assumption, Hypothesis 4. This final hypothesis is quite technical, but according to our analysis
presented in Section 5 it appears to be necessary. Moving forward, it would be interesting to investigate
situations in which this assumption can be weakened or eliminated all together, but at present there
does not appear to be a simpler way to obtain the main result Theorem 3.2.

With regards to weakening hypotheses, a quick comparison between Hypothesis 1 and the assump-
tions put on the Lambda-Omega lattice dynamical system studied in [2] reveals a slight, but important,
difference in the form of the function \( \omega \). That is, here we have assumed that \( \partial_\omega (a, \alpha) = 0 \), whereas
this was not a necessary assumption to obtain rotating wave solutions to system (3.1). Hence, it would
be of interest to determine whether the class of functions \( \omega \) for which stability results analogous to
Theorem 3.2 can be obtained can be expanded beyond that which is studied in this work. Interestingly,
it is the form of the functions \( \omega \) studied in this manuscript that is much more closely aligned with the
proof of the existence of rotating waves undertaken in the continuous spatial setting [8], making the work
[2] a more general investigation. It should be noted by the reader that the assumption \( \partial_\omega (a, \alpha) = 0 \)
played an integral role in providing the estimate in Lemma 5.8 which gave way to the main nonlinear
stability result. Hence, the specific class of the functions \( \omega \) studied in this manuscript is sufficient for
obtaining local asymptotic stability of periodic solutions to (3.1), but a potentially equally interesting
problem would be determining which functions \( \omega \) are necessary for obtaining stability.

Another point to consider is that all initial conditions are taken from the Banach space \( \ell^1 \). The reason
for this is that Hypothesis 3 is stated in terms of initial conditions in \( \ell^1 \), which as previously mentioned,
comes from the investigation [3]. It would be interesting to investigate how the results of Theorem 3.2
change as one considers initial conditions in \( \ell^p \), for various \( p > 1 \). What can be stated immediately is
that the stability results fail when considering the full range of initial conditions in \( \ell^\infty \). This is quite easy
to see since \( \ell^\infty \) contains the constant sequences indexed by \( \mathbb{Z}^2 \). To see this, denote \( \mathbf{1} = \{1\}_{(i,j)\in\mathbb{Z}^2} \in \ell^\infty \)
to be the constant sequence of all ones in \( \ell^\infty \) and note that for all \( C \in \mathbb{R} \) we have that \( (\mathbf{1}, \theta(\alpha) + C\mathbf{1}) \)
is also a steady-state solution of (3.7) due to the fact that only the difference of nearest-neighbours in \( \theta \)
are present in the polar system. Hence, taking the initial condition \((s^0, \psi^0) = (0, C\mathbf{1}) \in \ell^\infty \times \ell^\infty \) results
in \( \dot{s} = 0 \) and \( \dot{\psi} = 0 \) for all \( t \geq 0 \), \( C \in \mathbb{R} \) and \( \alpha \in [0, \alpha^*] \), thus giving no decay back to the equilibrium
\((s, \psi) = (0, 0) \), regardless of how small \( |C| \) is chosen. Therefore, it would be of great interest to see if the
decay rates gradually weaken for initial conditions belonging to \( \ell^p \) as \( p \) increases, resulting in no decay
in the limiting case of \( p = \infty \).
This work merely provides a necessary initial step in the study of stability in lattice dynamical systems through the use of traditional dynamical systems techniques. As one can see from this section and the two previous sections, there still remains a number of open problems related to this work. Away from the Lambda-Omega systems investigated here, the nonlinear stability of solutions to lattice dynamical systems remains one that is largely unexplored without the use of comparison theorems. Therefore, the hypotheses and techniques put forth in this manuscript could also be taken to inform future investigations into the local asymptotic stability of solutions to lattice dynamical systems.

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References


