The Swift-Hohenberg Equation

Spatially-localized structures occur in the natural world, such as in vegetation patterns, crime hotspots, and ferrofluids. The Swift-Hohenberg equation is a widely studied nonlinear partial differential equation that can describe many spatially localized structures. Radially-symmetric solutions to the Swift-Hohenberg equation in $n$-dimensions satisfy the partial differential equation to (1) for $n = 2$:

$$u_t = - \left( \frac{n - 1}{r} \partial_r + \partial_t \right)^2 u - \mu u + 2u^3 - u^4,$$

where $u = u(r, t)$, $r := |x|$, $x \in \mathbb{R}^n$, and $\mu$ is a bifurcation parameter.

The dimension of the underlying space, $n$, enters explicitly into equation (1). The one-dimensional equation therefore exhibits significantly different properties from the higher-dimensional Swift-Hohenberg equations:

- One Dimensional Equation:
  - Autonomous
  - Non-Singular
  - Hamiltonian

- Higher Dimensional Equations:
  - Non-Autonomous
  - Singular at $r = 0$
  - Not Hamiltonian

Snaking Bifurcations

In one spatial dimension ($n = 1$) equation (1) possesses spatially localized pulse steady-states which exhibit a bifurcation phenomena known as *snaking* [1].

- Solutions of the form shown to the left bounces between two different values of the parameter $\mu$, while ascending in the $L^2$-norm by simply adding another roll to the front of the wave train.
- It is known that these pulses come in pairs: one with a maximum at $r = 0$ and another with a minimum at $r = 0$.
- Bifurcation diagram resembles two intertwined snakes which ascend vertically in an unbounded manner.

Snaking in Higher Dimensions

Moving to higher spatial dimensions ($n = 2, 3$) the bifurcation structure of the pulse steady-state solutions splits into three distinct components: a lower snaking branch, isolas, and an upper snaking branch.

Lower Snaking Branch
- Bifurcation behaviour analogous to 1D equation
- Only extends vertically to finite height

Isolas
- Collection of closed curves
- Start after maximum height of lower branch
- Only extend vertically to finite height

Upper Snaking Branch
- Start after maximum height of isolas
- Rolls are added from the back at $r = 0$
- Conjectured to extend infinitely in the vertical direction

Spatial Dynamics

Steady-state solutions of (1) with this dimensional perturbation then satisfy

$$0 = - \left( 1 + \frac{\varepsilon}{r} \partial_r \right)^2 u - \mu u + 2u^3 - u^4,$$

which is now a fourth-order ordinary differential equation. Letting $u_1 = u$, $u_2 = \partial_r u$, $u_3 = (1 + \frac{\varepsilon}{r} \partial_r) u$ and $u_4 = \partial_r u_4$, we can consider the equivalent first order system

$$\begin{align*}
(u_1)_t &= u_2, \\
(u_2)_t &= u_3 - u_1 - \frac{\varepsilon}{r} u_2, \\
(u_3)_t &= u_4, \\
(u_4)_t &= - u_3 - \mu u_1 + 2u_1^2 - u_2^2 - \frac{\varepsilon}{r} u_4.
\end{align*}$$

Open Problems

- What causes the lower branch to have finite height and why does it behave similar to 1D snaking?
- What drives the formation of the isolas and the upper snaking branch?
- Are the isolas and upper snaking branch unique to the Swift-Hohenberg equation, or should they be expected when moving to higher spatial dimensions in other reaction-diffusion type equations which exhibit snaking in 1D?

Dimensional Perturbation

To understand the higher dimensional snaking cases, we focus on introducing a dimensional perturbation to equation (1) by considering $n := 1 + \varepsilon$, for small $\varepsilon > 0$. We are then able to use perturbative techniques to continuously vary $\varepsilon$ and inspect how the non-autonomous perturbation effects the snaking bifurcation curves.

Results

- We are able to show that for small $\varepsilon > 0$ the lower snaking branch is formed in a similar way to the one-dimensional snaking curves, and letting $L$ be this upper bound in $L^2$-norm, we find that it changes as a function of $\varepsilon$, and is approximately given by:
  $$L = C \varepsilon^4.$$

- In more general PDEs, we determine sufficient conditions for the lower snaking branch to have no upper bound based upon the flow in the direction of the energy.

- The formation of the isolas and upper snaking branch still remain an open topic of investigation which will be the subject of future work.

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References
